The goal of this course is to introduce you to the basic notions, theorems and open problems of the theory of the infinite dimensional representations of Lie groups. All this shall be explained in detail for the examples we shall consider, leaving the general approach for the future study of those who shall be interested. Needless to say, there are many connections between the Representation Theory and the rest of Mathematics. One aspect I would like to highlight is that unlike the finite dimensional theory, which tends to overlap with Algebraic Geometry, the infinite dimensional theory has its origins in Functional Analysis.

We shall work mainly in $L^2(\mathbb{R}^n)$, the space of the square integrable functions on the $n$-dimensional real vector space $\mathbb{R}^n$. The classical groups, such as the Heisenberg group, the symplectic and the orthogonal groups, act in a natural way on $L^2(\mathbb{R}^n)$. This leads to a decomposition of the Hilbert space $L^2(\mathbb{R}^n)$ into a direct sum, or direct integral, of irreducible subspaces.

In the process of decomposing the space $L^2(\mathbb{R}^n)$ we shall encounter some familiar and some less familiar names and objects, such as a Lie algebra, the Schwartz space, a Hermite function, the energy operator of the harmonic oscillator, a Harish-Chandra module and the Weil representation. Also, as we shall see, the Fourier transform corresponds to a group element. Thus the above mentioned decomposition of $L^2(\mathbb{R}^n)$ may be viewed as a study of the symmetry properties of the Fourier transform.

The Lie groups are real analytic manifolds, and each irreducible representation has a character which is a distribution on the group, in fact a locally integrable function. There is a one to one correspondence between the characters and the irreducible representations. In particular the character of an infinite dimensional representation can’t be a smooth function. This leads to a microlocal study of characters. We’ll take a look at it if the time allows.