HOWE'S CORRESPONDENCE FOR A GENERIC HARMONIC ANALYST

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1. Introduction

In the late seventies, the theory of pseudo-differential operators was a hot topic in Wrocław. Paweł Głowacki was finishing his PhD thesis, [Glo82]. There was a lively seminar frequented by advanced undergraduates, including the second author. One had to know, for example, that a pseudo-differential operator with symbol of order zero is bounded on $L^2(\mathbb{R}^n)$, [Hör85, Theorem 18.1.11]. Also, the symmetry properties of the Fourier transform, [SW71, Chapter 4] and the philosophy of the coadjoint orbits, [Kir62] were a must. This atmosphere of interest and excitement was created and sustained in a natural and seemingly effortless way, by Andrzej Hulanicki.

At the time and place, when the access to a Xerox machine was about as difficult as is the process of obtaining a pilot license and renting a private plane in Oklahoma, Andrzej distributed a few copies of Howe's paper, [How80], which ever since became one of our primary sources of information. That article, together with [How88], provides a construction of the oscillator representation suitable for "a generic harmonic analyst". We recall it in section 3.

In the following sections we introduce dual pairs and recall the definition of Howe's correspondence. Further we focus on some properties of the correspondence which might appeal to the reader we have in mind.

Anyone interested in the efforts to describe the correspondence in terms of Langlands parameters might consult [Pau98], [Pau00] and [AB95]. There are connections with particle physics [How85] and the theory of automorphic forms, [How79]. A good place to look for examples is [RH92].

2. Pseudo-differential Operators and the Wave Front Set of a Distribution

In this section we recall some basic notions and facts from the theory of the pseudo-differential operators following [Hör85, Chapter 18].

A linear partial differential operator of order m, with C^∞ coefficients, on an open set $X \subseteq \mathbb{R}^n$ is an expression of the form

$$\mathrm{P} = \sum_{|\alpha|\leq m} a_\alpha(x)\,\mathrm{D}^\alpha. \tag{1}$$

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Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n)$ is a multi-index with the $\alpha_j = 0, 1, 2, ...$, $|\alpha| = \alpha_1 + \alpha_2 + ... + \alpha_n$, each $a_\alpha \in C^\infty(X)$ and $\mathrm{D}^\alpha = \mathrm{D}_1^{\alpha_1}\mathrm{D}_1^{\alpha_2}...\mathrm{D}_1^{\alpha_n}$, where $\mathrm{D}_j = -i\frac{\partial}{\partial_j}$. In terms of the Fourier transform

$$\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-ix\cdot\xi} u(x)\,dx \qquad (u \in L^1(\mathbb{R}^n), \xi \in \mathbb{R}^n), \tag{2}$$

with $x \cdot \xi = x_1\xi_1 + x_2\xi_2 + ... + x_n\xi_n$, the action of P on a test function $u \in C_c^\infty(X)$ may be rewritten as

$$Pu(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix\cdot\xi} a(x,\xi)\,\hat{u}(\xi)\,d\xi, \tag{3}$$

where (with $\xi^\alpha = \xi_1^{\alpha_1}\xi_2^{\alpha_2}...\xi_n^{\alpha_n}$),

$$a(x,\xi) = \sum_{|\alpha|\le m} a_\alpha(x)\xi^\alpha \qquad (x \in X,\ \xi \in \mathbb{R}^n). \tag{4}$$

The principal symbol of P is defined as

$$\mathrm{P}_m(x,\xi) = \sum_{|\alpha|=m} a_\alpha(x)\xi^\alpha \qquad (x \in X,\ \xi \in \mathbb{R}^n). \tag{5}$$

For a real number m let $\mathrm{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$ be the space of all the functions $a \in C^\infty(\mathbb{R}^n \times \mathbb{R}^n)$ such that for all the multi-indices α and β, there are finite constants $C_{\alpha,\beta}$, so that

$$|D_\xi^\alpha D_x^\beta a(x,\xi)| \le C_{\alpha,\beta}(1 + |\xi|)^{m-|\alpha|} \qquad ((x,\xi) \in \mathbb{R}^n \times \mathbb{R}^n). \tag{6}$$

For each $a \in \mathrm{S}^m(\mathbb{R}^n \times \mathbb{R}^n)$, the formula (3) defines a continuous endomorphism of the Schwartz space $\mathcal{S}(\mathbb{R}^n)$, denoted by $\mathrm{Op}\,a$. This is an integral kernel operator

$$\mathrm{Op}\,a\,u(x) = \int_{\mathbb{R}^n} K(x,x')\,u(x')\,dx', \tag{7}$$

where

$$K(x,x') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} a(x,\xi)\,d\xi \qquad (x,x' \in \mathbb{R}^n). \tag{8}$$

Suppose X is an n-dimensional manifold with a C^∞ structure consisting of homeomorphisms κ of open set $X_\kappa \subseteq X$ onto open sets $\tilde{X}_\kappa \subseteq \mathbb{R}^n$ such that all the transition functions

$$\kappa'\kappa^{-1} : \kappa(X_\kappa \cap X_{\kappa'}) \to \kappa'(X_\kappa \cap X_{\kappa'})$$

are smooth and the patches X_κ cover X. A pseudo-differential operator of order $m \in \mathbb{R}$ on X is a continuous linear map $A : \mathbb{C}_c^\infty(X) \to C^\infty(X)$ such that for all the X_κ and all $\phi, \psi \in C_c^\infty(\tilde{X}_\kappa)$, the corresponding localization of the lift of A is given by the formula (3) where a satisfies (6). In other words,

$$\phi\left((\kappa^{-1})^* A\kappa^*\right)\psi \in \mathrm{Op}\,\mathrm{S}^m(\mathbb{R}^n \times \mathbb{R}^n). \tag{9}$$

We see from (3) that the operator (9) is an endomorphism of $\mathcal{S}(\mathbb{R}^n)$. Hence, A extends to a continuous linear map from the space of the compactly supported distributions on X to the distributions on X:

$$A : \mathcal{E}'(X) \to \mathcal{D}'(X). \tag{10}$$

Denote by $\Psi^m(X)$ the space of all the pseudo-differential operators of order $m \in \mathbb{R}$ on X.

Recall the cotangent bundle T^*X of X. Let $\mathrm{S}^m(T^*X) \subseteq C^\infty(T^*X)$ be the space of all the functions a such that for each coordinate patch X_κ, the pullback $\tilde{a}_\kappa$ of a to $T^*(\tilde{X}_\kappa) = \tilde{X}_\kappa \times \mathbb{R}^n$ is in $\mathrm{S}_{loc}(\tilde{X}_\kappa \times \mathbb{R}^n)$, i.e. $\phi(x)\tilde{a}_\kappa(x,\xi)$ is in $\mathrm{S}(\tilde{X}_\kappa \times \mathbb{R}^n)$ for any $\phi \in C_c^\infty(\tilde{X}_\kappa)$. One shows that for each $A \in \Psi^m(X)$ there is a unique $a_m \in \mathrm{S}^m(T^*X)$, modulo $\mathrm{S}^{m-1}(T^*X)$, such that for each X_κ the pullback of A to $\tilde{X}_\kappa$ is given by the formula (3) in terms of the function $\tilde{a}_{m,\kappa}$ plus an operator with a smooth integral kernel. This leads the to definition of the principal symbol

$$\Psi^m(X) \ni A \to a_m + \mathrm{S}^{m-1}(T^*X) \in \mathrm{S}^m(T^*X)/\mathrm{S}^{m-1}(T^*X). \tag{11}$$

In particular, if X is an open subset of $\mathbb{R}^n$ and $A = P$ is the differential operator (1) then one may choose $a_m = \mathrm{P}_m$ as in (5).

An operator $A \in \Psi^m(X)$ is said to be non-characteristic at the image of $(x_0, \xi_0) \in \tilde{X}_\kappa \times (\mathbb{R}^n \setminus 0)$ in $T^*X \setminus 0$ if there is an open neighborhood U of x_0 in $\tilde{X}_\kappa$, an open cone $V \subseteq \mathbb{R}^n \setminus 0$ containing ξ_0 and two constants c and C such that

$$|\tilde{a}_{m,\kappa}(x,\xi)| \geq c|\xi|^m \qquad (x \in U,\ \xi \in V,\ |\xi| > C). \tag{12}$$

A point of the cotangent bundle is called a characteristic point iff it is not a non-characteristic point. The set of the characteristic points of A is denoted by $Char\, A$. If X is an open subset of $\mathbb{R}^n$ and $A = P$ is the differential operator (1) then $Char P$ coincides with the set of the zeros of the principal symbol P_m.

Finally, A is said to be properly supported if for every compact subset $K \subseteq X$ there is another compact subset $K' \subseteq X$ such that

$$supp\, u \subseteq K \Rightarrow supp\, Au \subseteq K' \text{ and } u = 0 \text{ in } K' \Rightarrow Au = 0 \text{ in } K. \tag{13}$$

Under this condition A extends to a map

$$A : \mathcal{D}'(X) \to \mathcal{D}'(X). \tag{14}$$

For a distribution $u \in \mathcal{D}'(X)$ one may define the wave front set of u to be the intersection of the characteristic sets of all the properly supported pseudo-differential operators $A \in \Psi^m(X)$ such that $Au \in C^\infty(X)$:

$$WF(u) = \bigcap Char\, A \subseteq T^*X \setminus 0. \tag{15}$$

This definition does not depend on the order m. In what follows we shall distinguish between $\mathbb{R}^n$ and the dual, $\mathbb{R}^{n*}$, without the identification via the dot product we have used so far.

3. The oscillator representation of the metaplectic group

Here we recall Howe's construction of the oscillator representation, [How88], which is rooted in Weyl calculus of the pseudo-differential operators, where instead of (8) one considers operators with the integral kernel

$$K(x,x') = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{i(x-x')\cdot\xi} a((x+x')/2, \xi)\, d\xi. \tag{16}$$

Since we are heading towards group representation theory it is reasonable to switch to a coordinate free approach.

Let W be a vector space of dimension $2n < \infty$ over $\mathbb{R}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Denote by $\mathrm{Sp} = Sp(\mathsf{W}) \subseteq \mathrm{End}(\mathsf{W})$ the corresponding symplectic group, i.e. the group of the isometries of the form $\langle\ ,\ \rangle$, and by $\mathfrak{sp} = \mathfrak{sp}(\mathsf{W}) \subseteq \mathrm{End}(\mathsf{W})$ the Lie algebra of Sp. An element $\mathcal{J} \in \mathrm{End}(\mathsf{W})$ is called a compatible positive definite complex structure if

$$\begin{aligned} &\langle \mathcal{J}w, w'\rangle = \langle \mathcal{J}w', w\rangle && (w, w' \in \mathsf{W}),\\ &\langle \mathcal{J}w, w\rangle > 0 && (w \in \mathsf{W}\setminus 0),\\ &\mathcal{J}^2 = -\mathcal{I}, \end{aligned} \tag{17}$$

where $\mathcal{I}$ stands for the identity map on W. The first condition in (17) means that $\mathcal{J} \in \mathfrak{sp}$ and the two top condition say that $\langle \mathcal{J}\ ,\ \rangle$ defines a symmetric positive definite bilinear form on W. Such elements exist and form a single orbit under the conjugation action of Sp.

For an arbitrary vector subspace $\mathsf{U} \subseteq \mathsf{W}$ we normalize the Lebesgue measure du on U so that the volume of the unit cube with respect to the form $\langle \mathcal{J}\ ,\ \rangle$ is 1. (Since the determinant of any element of Sp is 1, the normalization of dw, $w \in \mathsf{W}$, does not depend on the particular choice of $\mathcal{J}$.) Multiplication by the Lebesgue measure defines an embedding of the Schwartz space on U into the space of the tempered distributions on U,

$$\mathcal{S}(\mathsf{U}) \to \mathcal{S}^*(\mathsf{U}), \tag{18}$$

which we shall use without comments.

Fix the unitary character $\chi(r) = e^{2\pi i r}$, $r \in \mathbb{R}$. Pick a complete polarization

$$\mathsf{W} = \mathsf{X} \oplus \mathsf{Y} \tag{19}$$

(Here X and Y are maximal isotropic subspaces of W.) Recall the Weyl transform

$$\begin{aligned} &\mathcal{K} : S^*(\mathsf{W}) \to S^*(\mathsf{X}\times\mathsf{X}), \\ &\mathcal{K}(f)(x, x') = \int_{\mathsf{Y}} f(x - x' + y)\chi(\frac{1}{2}\langle y, x + x'\rangle)\, dy, \end{aligned} \tag{20}$$

with the inverse given by

$$\mathcal{K}(K)(x + y) = 2^{-n} \int_{\mathsf{X}} K(\frac{x'+x}{2}, \frac{x'-x}{2})\chi(\frac{1}{2}\langle x', y\rangle)\, dx' \qquad (x, x' \in \mathsf{X},\ y \in \mathsf{Y}). \tag{21}$$

Each element $K \in S^*(\mathsf{X}\times\mathsf{X})$ defines an operator $Op(K) \in \mathrm{Hom}(S(\mathsf{X}), S^*(\mathsf{X}))$ by

$$Op(K)v(x) = \int_{\mathsf{X}} K(x, x')v(x')\, dx'. \tag{22}$$

The map Op extends to an isomorphism of linear topological spaces $\mathcal{S}^*(\mathsf{X}\times\mathsf{X})$ and $\mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X}))$. This is known as the Schwartz Kernel Theorem, [Hör83, Theorem 5.2.1]. For suitable K, the operator $Op(K)$ is of trace class and

$$tr\, Op(K) = \int_{\mathsf{X}} K(x, x)\, dx. \tag{23}$$

For $\phi_1, \phi_2 \in \mathcal{S}(\mathsf{W})$, we have the twisted convolution,

$$\phi_1 \natural \phi_2(w') = \int_{\mathsf{W}} \phi_1(w)\phi_2(w'-w)\chi(\frac{1}{2}\langle w, w'\rangle)\, dw \qquad (w' \in \mathsf{W}), \tag{24}$$

which extends to some, but not all, tempered distributions. Fourier inversion formula shows that the map

$$\rho = Op \circ \mathcal{K} : \mathcal{S}(\mathsf{W}) \to \mathrm{Hom}(\mathcal{S}(\mathsf{X}), \mathcal{S}^*(\mathsf{X})) \tag{25}$$

is an isomorphism of linear topological vector spaces with

$$tr(\rho(f)) = f(0) \tag{26}$$

for suitable $f \in \mathcal{S}^*(\mathsf{W})$.

Recall the Cayley transform $c(y) = (y+1)(y-1)^{-1}$. This is a rational map from $\mathrm{End}(\mathsf{W})$ to itself, which restricts to a birational isomorphism of Sp and $\mathfrak{sp}$. Let

$$\widetilde{\mathrm{Sp}}^c = \{\tilde{g} = (g, \xi) \in \mathrm{Sp} \times \mathbb{C},\ det(g-1) \neq 0,\ \xi^2 = det(i(g-1))^{-1}\}. \tag{27}$$

For each $x \in \mathfrak{sp}$, $\langle x\ ,\ \rangle$ is a symmetric bilinear form on W with the signature $sgn\langle x\ ,\ \rangle$ equal to the maximal dimension of a subspace where this form is positive definite minus the maximal dimension of a subspace where this form is negative definite. Set

$$chc(x) = 2^n |det(x)|^{-\frac{1}{2}}\, exp(\frac{\pi}{4} i\, sgn\langle x\ ,\ \rangle) \qquad (x \in \mathfrak{sp},\ det(x) \neq 0). \tag{28}$$

(This is a Fourier transform of one of the two minimal non-zero nilpotent co-adjoint orbits in $\mathfrak{sp}^*$, [Prz00, Proposition 9.3] and [Hör83, Theorem 7.6.1].) For two elements $(g_1, \xi_1), (g_2, \xi_2) \in \widetilde{\mathrm{Sp}}^c$, with $c(g_1) + c(g_2)$ invertible, define a product

$$(g_1, \xi_1)(g_2, \xi_2) = (g_1 g_2, \xi_1 \xi_2 chc(c(g_1) + c(g_2)), \tag{29}$$

and let

$$\begin{aligned} \Theta &: \widetilde{\mathrm{Sp}}^c \ni \tilde{g} = (g, \xi) \to \xi \in \mathbb{C}, \\ T &: \widetilde{\mathrm{Sp}}^c \ni \tilde{g} \to \Theta(\tilde{g})\chi_{c(g)} \in S^*(\mathsf{W}), \end{aligned} \tag{30}$$

where, for $x = c(g) \in \mathfrak{sp}$, $\chi_x(w) = \chi(\frac{1}{4}\langle x(w), w\rangle)$.

Theorem 1. *Up to a group isomorphism there is a unique connected group* $\widetilde{\mathrm{Sp}}$ *containing* $\widetilde{\mathrm{Sp}}^c$ *with the multiplication given by (29) on the indicated subset of* $\widetilde{\mathrm{Sp}}^c \times \widetilde{\mathrm{Sp}}^c$. *The map*

$$\widetilde{\mathrm{Sp}}^c \ni \tilde{g} \to g \in \mathrm{Sp}$$

extends to a double covering homomorphism

$$\widetilde{\mathrm{Sp}} \ni \tilde{g} \to g \in \mathrm{Sp}. \tag{31}$$

The map T extends to a continuous injection $T : \widetilde{\mathrm{Sp}} \to S^*(\mathsf{W})$ *and*

$$T(\tilde{g}_1)\natural T(\tilde{g}_2) = T(\tilde{g}_1\tilde{g}_2) \qquad (\tilde{g}_1, \tilde{g}_2 \in \widetilde{\mathrm{Sp}}^c,\ det(c(g_1) + c(g_2)) \neq 0), \tag{32}$$

$$T(1) = \delta, \tag{33}$$

where δ is the Dirac delta at the origin.

The composition $\omega = \rho \circ T$ *maps* $\widetilde{\mathrm{Sp}}$ *into the group of the unitary operators on the Hilbert space* $L^2(\mathsf{X})$*. The operators* $\omega(g)$*,* $g \in \widetilde{\mathrm{Sp}}$*, preserve the subspace* $\mathcal{S}(\mathsf{X}) \subseteq L^2(\mathsf{X})$ *and extend uniquely to endomorpisms of* $\mathcal{S}^*(\mathsf{X})$*.*

Moreover, Θ *is the distribution character of* ω *in the sense that for any* $\phi \in C_c^\infty(\widetilde{\mathrm{Sp}})$*,*

$$\omega(\phi) = \int_{\widetilde{\mathrm{Sp}}} \omega(g)\phi(g)\, dg$$

is of trace class and

$$tr\omega(\phi) = \int_{\widetilde{\mathrm{Sp}}} \Theta(g)\phi(g)\, dg,$$

where the last integral is absolutely convergent.

The title of this section refers to the unitary representation ω of the group $\widetilde{\mathrm{Sp}}$ on the Hilbert space $L^2(\mathsf{X})$. The representation ω does not factor to a representation of the symplecitc group, hence it is necessary to consider the double cover (31).

4. Dual pairs

A real reductive dual pair is a pair of subgroups $\mathrm{G}, \mathrm{G}' \subseteq \mathrm{Sp}(\mathsf{W})$ which act reductively on the symplectic space W, G' is the centralizer of G in Sp and G is the centralizer of G' in Sp, [How79]. We shall be concerned with the irreducible pairs in the sense that there is no non-trivial direct sum decomposition of W preserved by G and G'. For brevity we shall simply call them dual pairs.

There are two kinds of such pairs. Either G, G' coincides with $\mathrm{GL}_n(\mathbb{D}), \mathrm{GL}_m(\mathbb{D})$, with $\mathbb{D} = \mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ - the quaternions, or there is a (possibly trivial) involution ι of $\mathbb{D}$ over $\mathbb{R}$ so that G, G' are isomorphic to a pair of the isometry groups of two non-degenerate hermitian forms over $\mathbb{D}$ of opposite type.

Specifically, in the first case, there are two finite dimensional left vector spaces, V, V' over $\mathbb{D}$,

$$\mathsf{W} = \mathrm{Hom}(\mathsf{V}, \mathsf{V}') \oplus \mathrm{Hom}(\mathsf{V}', \mathsf{V}), \tag{34}$$

viewed as a vector space over $\mathbb{R}$, and

$$\langle (A, B), (A', B') \rangle = tr_{\mathbb{D}/\mathbb{R}}(AB') - tr_{\mathbb{D}/\mathbb{R}}(BA'), \tag{35}$$

where $A, A' \in \mathrm{Hom}(\mathsf{V}, \mathsf{V}')$, $B, B' \in \mathrm{Hom}(\mathsf{V}', \mathsf{V})$ and $tr_{\mathbb{D}/\mathbb{R}}(C)$ is the trace of C viewed as a linear map over $\mathbb{R} \subseteq \mathbb{D}$. The groups $\mathrm{GL}(\mathsf{V})$, $\mathrm{GL}(\mathsf{V}')$ act on W by

$$g(A, B) = (Ag^{-1}, gB),\ g'(A, B) = (g'A, Bg'^{-1}) \qquad (g \in \mathrm{GL}(\mathsf{V}), g' \in \mathrm{GL}(\mathsf{V}')), \tag{36}$$

This action preserves the form (35) and gives the desired embedding of $\mathrm{GL}(\mathsf{V})$, $\mathrm{GL}(\mathsf{V}')$ into $\mathrm{Sp}(\mathsf{W})$.

In the second case there are two non-degenerate forms $(\ ,\ )$, $(\ ,\ )'$ on V, V' respectively,

$$\mathsf{W} = \mathrm{Hom}(\mathsf{V}, \mathsf{V}'), \tag{37}$$

viewed as a vector space over $\mathbb{R}$, and in terms of the following map

$$\mathrm{Hom}(\mathsf{V}, \mathsf{V}') \ni w \to w^* \in \mathrm{Hom}(\mathsf{V}', \mathsf{V}),\ \ (wv, v')' = (v, w^*v') \qquad (v \in \mathsf{V},\ v' \in \mathsf{V}'), \tag{38}$$

the symplectic form is defined by

$$\langle w, w'\rangle = tr_{\mathbb{D}/\mathbb{R}}(w'^{*}w) \qquad (w, w' \in \mathsf{W}). \tag{39}$$

The isometry groups $G(\mathsf{V}, (\ ,\ ))$, $G(\mathsf{V}', (\ ,\ )')$ act on W by

$$g(w) = wg^{-1},\ g'(w) = g'w, \qquad (g \in G(\mathsf{V}, (\ ,\ )),\ g' \in G(\mathsf{V}', (\ ,\ )),\ w \in \mathsf{W}), \tag{40}$$

preserve the form (39) and consequently may be identified with the corresponding subgroups of $\mathrm{Sp}(\mathsf{W})$. We collect the dual pairs in the following table,

Dual pair	division algebra	involution ι	type of forms	$dim\,\mathsf{W}$
$\mathrm{GL}_n(\mathbb{D}),\ \mathrm{GL}_m(\mathbb{D})$	$\mathbb{R},\ \mathbb{C},\ \mathbb{H}$			$2nm\,dim_{\mathbb{R}}(\mathbb{D})$
$\mathrm{O}_{p,q},\ \mathrm{Sp}_{2n}(\mathbb{R})$	$\mathbb{R}$	trivial	orthogonal, symplectic	$2n(p+q)$
$\mathrm{Sp}_{2n}(\mathbb{R}),\ \mathrm{O}_{p,q}$	$\mathbb{R}$	trivial	symplectic, orthogonal	$2n(p+q)$
$\mathrm{O}_p(\mathbb{C}),\ \mathrm{Sp}_{2n}(\mathbb{C})$	$\mathbb{C}$	trivial	orthogonal, symplectic	$4np$
$\mathrm{Sp}_{2n}(\mathbb{C}),\ \mathrm{O}_p(\mathbb{C})$	$\mathbb{C}$	trivial	symplectic, orthogonal	$4np$
$\mathrm{U}_{p,q},\ \mathrm{U}_{r,s}$	$\mathbb{C}$	non-trivial	hermitian, skew-hermitian	$2(p+q)(r+s)$
$\mathrm{Sp}_{p,q},\ \mathrm{O}^*_{2n}$	$\mathbb{H}$	non-trivial	hermitian, skew-hermitian	$8n(p+q)$
$\mathrm{O}^*_{2n},\ \mathrm{Sp}_{p,q}$	$\mathbb{H}$	non-trivial	skew-hermitian, hermitian	$8n(p+q)$

5. The correspondence for dual pairs with one member compact

Recall that a unitary representation of a topological group G on a Hilbert space H (with a countable basis) is a group homomorphism Π from G into the group of the unitary operators on H such that for all $v \in \mathsf{H}$ the map

$$\mathrm{G} \ni g \to \Pi(g)v \in \mathsf{H}$$

is continuous. Our basic example is $\mathrm{G} = \widetilde{\mathrm{Sp}}$, $\Pi = \omega$ and $\mathsf{H} = \mathrm{L}^2(X)$.

Two unitary representations (Π_1, H_1), (Π_2, H_2) are equivalent iff there is a bijective isometry $T : \mathsf{H}_1 \to \mathsf{H}_2$ such that

$$\Pi_2(g)T = T\Pi_1(g) \qquad (g \in \mathrm{G}). \tag{41}$$

The representation (Π, H) is irreducible iff the only $\Pi(\mathrm{G})$-closed invariant subspaces are H and $\{0\}$. The set of the equivalence classes of the irreducible unitary representations of G is usually denoted by $\hat{\mathrm{G}}$.

If G is compact, then any unitary representation (Π, H) decomposes into the Hilbert direct sum of irreducible representations, [HR63, Theorem 27.44]. By grouping together the mutually isomorphic representations we obtain the decomposition of (Π, H) into the isotypic components $(\Pi, \mathsf{H}_{\Pi_1})$:

$$\mathsf{H} = \bigoplus \mathsf{H}_{\Pi_1}. \tag{42}$$

(Here the summation is over a subset of $\hat{\mathrm{G}}$.)

In particular if G is a compact member of a dual pair, then the preimage $\widetilde{\mathrm{G}}$ of G in $\widetilde{\mathrm{Sp}}$ is also compact and

$$L^2(X) = \bigoplus L^2(X)_{\Pi}, \tag{43}$$

where the summation is over certain subset of $\hat{\widetilde{\mathrm{G}}}$. Let us denote this subset by $R(\widetilde{\mathrm{G}}, \omega)$.

Since the operators $\omega(g)$, $\omega(g')$, $(g \in \widetilde{\mathrm{G}},\ g' \in \widetilde{\mathrm{G}}')$, commute, $\omega(\widetilde{\mathrm{G}}')$ preserves each subspace $L^2(X)_\Pi$. This way $L^2(X)_\Pi$ becomes a unitary representation of $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}}'$. The point is that this representation is irreducible, [How89a]. As such, it is isomorphic to $\Pi \otimes \Pi'$ for some $\Pi' \in \hat{\widetilde{\mathrm{G}}}'$. Let us denote the set of the equivalence classes of the resulting Π''s by $\mathcal{R}(\widetilde{\mathrm{G}}', \omega)$. Furthermore, different Π's yield different Π''s, [How89a]. The resulting bijection

$$\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega), \tag{44}$$

is the Howe's correspondence in this case.

The decomposition (43) for the pair O_2, $\mathrm{Sp}_2(\mathbb{R})$ occurs, for example, in [SW71, sec. 4.1], and is used to express the Fourier transform in terms of Bessel functions. For more information on such connections see [RH92]. An explicit description of the decomposition (43) in terms of highest weights is available in [KV78]. With very few exceptions the representations which occur in (43) exhaust the so called unitary highest weight representations of the groups involved, see [EHW83].

6. The correspondence for a general dual pair

So far we did not have to refer to any results of Harish-Chandra, but this is about to change.

Let V be a linear topological vector space and let G be a topological group. A representation of G on V is a group homomorphism Π from G into the group of the continuous invertible endomorphisms of V such that for each $v \in \mathsf{V}$, the map

$$\mathrm{G} \ni g \to \Pi(g)v \in \mathsf{V} \tag{45}$$

is continuous.

Suppose G is a Lie group. Then $\mathsf{V}^\infty \subseteq \mathsf{V}$ is defined to be the subspace of all the $v \in \mathsf{V}$ such that the maps are (45) are smooth. For example, if $\mathrm{G} = \widetilde{\mathrm{Sp}}$ and $\mathsf{V} = \mathrm{L}^2(X)$ or $\mathcal{S}^*(X)$ (with the weak* topology), then $\mathsf{V}^\infty = \mathcal{S}(X)$.

The Lie algebra $\mathfrak{g}$ of the Lie group G acts on V^∞ by

$$\Pi(x)v = \frac{d}{dt}\Pi(exp(tx))v|_{t=0} \qquad (x \in \mathfrak{g},\ v \in \mathsf{V}^\infty). \tag{46}$$

This action extends uniquely to an action of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of the Lie algebra $\mathfrak{g}$. Let $\mathcal{U}(\mathfrak{g})^{\mathrm{G}}$ denote the subalgebra of the G invariants in $\mathcal{U}(\mathfrak{g})$. This may be a proper subalgebra of the center of $\mathcal{U}(\mathfrak{g})$ if the group G is disconnected. One may think of $\mathcal{U}(\mathfrak{g})^{\mathrm{G}}$ as of the algebra of the left and right invariant differential operators on G.

The representation (Π, V) is called quasisimple if there is an algebra homomorphism

$$\gamma_\Pi : \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \to \mathbb{C} \tag{47}$$

such that

$$\Pi(z)v = \gamma_\Pi(z)v \qquad (z \in \mathcal{U}(\mathfrak{g})^{\mathrm{G}}, v \in \mathsf{V}^\infty). \tag{48}$$

Then γ_Π is called the infinitesimal character of Π.

Suppose G is a real reductive group, [Wal88]. (Any member of a dual pair or a finite cover of it is such a group.) Fix a maximal compact subgroup $\mathrm{K} \subseteq \mathrm{G}$. Let $\mathsf{V}^o \subseteq \mathsf{V}^\infty$ be the subspace of all the vectors v such that the linear span of $\Pi(K)v$ is finite dimensional. Then V^o is a $\mathfrak{g}$-module and a representation of K. The following conditions hold:
(a) V^o is a direct sum of subspaces invariant and irreducible under the action of K;
(b) the differential of the K action coincides with the one obtained from the inclusion $\mathfrak{k} \subseteq \mathfrak{g}$;
(c) the action of K on $\mathrm{End}(\mathsf{V}^o)$ by conjugation preserves the image of $\mathfrak{g}$ in $\mathrm{End}(\mathsf{V}^o)$ and coincides with the action of K on $\mathfrak{g}$. One refers to V^o as to a $(\mathfrak{g}, \mathrm{K})$-module, or the Harish-Chandra module of (Π, V).

Two representation (Π_1, V_1), (Π_2, V_2) are called infinitesimally equivalent iff the corresponding $(\mathfrak{g}, \mathrm{K})$-modules (as defined above) are isomorphic. The representation (Π, V) is called admissible iff each K-isotypic component of V^o is finite dimensional. These notions were introduced by Harish-Chandra in [Har51a] (for a connected group G). In particular he proved that any irreducible unitary representation of G is admissible, [Har53, Theorem 7], and that two irreducible unitary representations are equivalent if and only if they are infinitesimally equivalent, [Har53, Theorem 8]. Thus, without running into any contradictions, we may define $\mathcal{R}(\mathrm{G})$ to be the set of the infinitesimal equivalence classes of the irreducible admissible representations of G and consider $\hat{\mathrm{G}}$ to be a subset of $\mathcal{R}(\mathrm{G})$.

At this point it seems reasonable to mention that, thanks to the works of R. Langlands, A. Knapp and G. Zuckerman, (see [Wal88] for a nice exposition and references) there is a description of the set $\mathcal{R}(\mathrm{G})$ in terms of the so called Langlands' parameters, but with a few exceptions ([Bar89], [Vog86], [Vog94]) the location of $\hat{\mathrm{G}}$ inside $\mathcal{R}(\mathrm{G})$ is not known.

For a member G of a dual pair, let $\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \subseteq \mathcal{R}(\widetilde{\mathrm{G}})$ denote the subset of the representations which may be realized as quotients of $\mathcal{S}(X)$ by closed $\widetilde{\mathrm{G}}$-invariant subspaces. Let us fix a representation Π in $\mathcal{R}(\widetilde{\mathrm{G}}, \omega)$ and let $N_\Pi \subseteq \mathcal{S}(X)$ be the intersection of all the closed G-invariant subspaces $N \subseteq \mathcal{S}(X)$ such that Π is infinitesimally equivalent to $\mathcal{S}(X)/N$. This is a representation of both $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$. As such, it is infinitesimally isomorphic to

$$\Pi \otimes \Pi_1', \tag{49}$$

for some representation Π_1' of $\widetilde{\mathrm{G}}'$. Howe proved, [How89b, Theorem 1A], that Π_1' is a finitely generated admissible quasisimple representation of $\widetilde{\mathrm{G}}'$, which has a unique irreducible quotient $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$. Conversely, starting with $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$ and applying the above procedure with the roles of G and G′ reversed, we arrive at the representation $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$. The resulting bijection

$$\mathcal{R}(\widetilde{\mathrm{G}}, \omega) \ni \Pi \to \Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega) \tag{50}$$

is called Howe's correspondence, or local θ correspondence, for the pair G, G′.

Furthermore, let $\mathcal{R}(\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}', \omega)$ denote the set of the representations of $\widetilde{\mathrm{G}} \times \widetilde{\mathrm{G}}'$ which may be realized as quotients of $\mathcal{S}(X)$ by closed $\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'$-invariant subspaces. Then $\mathcal{R}(\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}', \omega)$ coincides with the bijection (50) in the sense that it consists of the tensor products $\Pi \otimes \Pi'$

where Π and Π' occur in (50), [How89b, Theorem 1]. Moreover, each such $\Pi \otimes \Pi'$ may be realized as a quotient of $\mathcal{S}(X)$ in a unique way, i.e. $dim\ \mathrm{Hom}_{\widetilde{\mathrm{G}}\widetilde{\mathrm{G}}'}(\mathcal{S}(X), \Pi \otimes \Pi') = 1$.

The space of the tempered distributions, $\mathcal{S}^*(X)$, comes equipped with the weak* topology. Therefore the space of the continuous linear functionals on $\mathcal{S}^*(X)$ coincides with $\mathcal{S}(X)$, [Rud91, Theorem 3.10]. For $N \subseteq \mathcal{S}(X)$ let $N^\perp \subseteq \mathcal{S}^*(X)$ be the annihilator of N, and for $R \subseteq \mathcal{S}^*(X)$ let $R^\perp \subseteq \mathcal{S}(X)$ be the annihilator of R. The Hahn-Banach Theorem, [Rud91, Theorem 3.7], implies that $N^{\perp\perp} = N$ and $R^{\perp\perp} = R$. Furthermore, the dual $(\mathcal{S}(X)/N)^* = N^\perp$.

Let $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$ be realized on $\mathcal{S}(X)/N$. Then the contragredient of Π^c of Π is realized on $(\mathcal{S}(X)/N)^*$, which in turn is isomorphic to $N^\perp \subseteq \mathcal{S}^*(X)$. Thus the contragredient of Π may be realized as a subrepresentation of the space of the tempered distributions. Similarly, the contragredient of Π' and $\Pi \otimes \Pi'$ may be realized as a subrepresentation of the space of $\mathcal{S}^*(X)$.

Thus instead of talking of the quotients we might equivalently use subrepresentations to describe the correspondence (50) with the Π and Π' replaced by the contragredients, as follows.

Let $\mathcal{R}^c(\widetilde{\mathrm{G}}, \omega) \subseteq \mathcal{R}(\widetilde{\mathrm{G}})$ denote the subset of the representations which may be realized as subrepresentations of $\mathcal{S}^*(X)$ on closed $\widetilde{\mathrm{G}}$-invariant subspaces. Let us fix a representation π in $\mathcal{R}^c(\widetilde{\mathrm{G}}, \omega)$ and let $R_\pi \subseteq \mathcal{S}^*(X)$ be the sum of all the closed G-invariant subspaces $R \subseteq \mathcal{S}^*(X)$ such that π is infinitesimally equivalent to R. This is a representation of both $\widetilde{\mathrm{G}}$ and $\widetilde{\mathrm{G}}'$. As such, it is infinitesimally isomorphic to $\pi \otimes \pi_1'$ for some representation π_1' of $\widetilde{\mathrm{G}}'$. Howe's Theorem says that π_1' has a unique irreducible subrepresentation $\pi' \in \mathcal{R}^c(\widetilde{\mathrm{G}}', \omega)$.

7. The correspondence of infinitesimal characters

Consider a real reductive group G and a Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$ with the Lie algebra $\mathfrak{h} \subseteq \mathfrak{g}$ and the subset of the regular elements $\mathrm{H}^{reg} \subseteq \mathrm{H}$. Let $\pi_{\mathrm{G}/\mathrm{H}}$ denote any analytic square root of the map

$$\mathrm{H}^{reg} \ni h \to det(Ad(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}} \in \mathbb{C}. \tag{51}$$

Recall the algebra $\mathcal{U}(\mathfrak{g})^{\mathrm{G}}$ of the left and right invariant differential operators on G. There is a bijective algebra isomorphism

$$\gamma_{\mathfrak{g}/\mathfrak{h}} : \mathcal{U}(\mathfrak{g})^{\mathrm{G}} \to \mathcal{U}(\mathfrak{h})^W, \tag{52}$$

where W is the corresponding Weyl group, such that for any test function $f \in C_c^\infty(\mathrm{G})$ and any $z \in \mathcal{U}(\mathfrak{g})^{\mathrm{G}}$

$$(z\,f)|_{\mathrm{H}^{reg}} = \frac{1}{\pi_{\mathrm{G}/\mathrm{H}}} \gamma_{\mathfrak{g}/\mathfrak{h}}(z) \left(\pi_{\mathrm{G}/\mathrm{H}} \cdot f|_{\mathrm{H}^{reg}}\right), \tag{53}$$

see [Har56, Theorem 2] and [Har63, Lemma 13]. (Here $|_{\mathrm{H}^{reg}}$ stands for the restriction to H^{reg}.)

For a dual pair G, G′ we shall define certain unnormalized moment maps

$$\tau : \mathsf{W} \to \mathfrak{g}, \quad \tau' : \mathsf{W} \to \mathfrak{g}' \tag{54}$$

as follows. If W is as in (34), then

$$\tau((A,B)) = BA, \quad \tau'((A,B)) = AB.$$

If W is as in (37), then

$$\tau(w) = w^* w, \quad \tau'(w) = ww^*.$$

These map intertwine the GG' action on W with the adjoint action on the corresponding Lie algebras:

$$\tau(gg'(w)) = Ad(g)(\tau(w)),\ \tau'(gg'(w)) = Ad(g')(\tau'(w)) \quad (g \in \mathrm{G},\ g' \in \mathrm{G}',\ w \in \mathsf{W}). \tag{55}$$

Notice that we may view any element $w \in \mathsf{W}$ as an endomorphism of $\mathsf{V} \oplus \mathsf{V}'$. Indeed, if we are in the case (34), let

$$(A,B)(v,v') = (Bv', Av) \qquad (A \in \mathrm{Hom}(V,V'),\ B \in \mathrm{Hom}(V',V),\ v \in \mathsf{V},\ v' \in \mathsf{V}').$$

If we are in the case (37), let

$$w(v,v') = (w^* v', wv) \qquad (w \in \mathsf{W},\ v \in \mathsf{V},\ v' \in \mathsf{V}').$$

We shall say that an element $w \in \mathsf{W}$ is semisimple if it is semisimple as an endomorphism of $\mathsf{V} \oplus \mathsf{V}'$. There is one dual pair O_1, Sp_{2n} (over $\mathbb{R}$ or $\mathbb{C}$), where the only semisimple element of W is zero. For the moment, let us exclude this case from our considerations. Then there is a finite collection $\mathfrak{w}_1$, $\mathfrak{w}_2$, ..., $\mathfrak{w}_m$, of vector subspaces of W such that any semisimple GG'-orbit in W passes through one of the $\mathfrak{w}_j$ and the distinct $\mathfrak{w}_j$ are not GG'-conjugate. They are called Cartan subspaces in [Prz06]. Fix one such Cartan subspace $\mathfrak{w} \subseteq \mathsf{W}$ and define $\mathrm{lin}\tau(\mathfrak{w})$ to be the linear span of $\tau(\mathfrak{w})$ and similarly for τ'. Then the relation

$$\{(\tau(w), \tau'(w));\ w \in \mathfrak{w}\} \subseteq \mathrm{lin}\tau(\mathfrak{w}) \times \mathrm{lin}\tau'(\mathfrak{w}) \tag{56}$$

is an invertible function which extends to a linear bijection

$$\mathrm{lin}\tau(\mathfrak{w}) \to \mathrm{lin}\tau'(\mathfrak{w}), \tag{57}$$

see [Prz06].

Suppose the rank of G (the dimension of any Cartan subalgebra of $\mathfrak{g}$) is less then or equal to the rank of G'. Then $\mathfrak{h} = \mathrm{lin}\tau(\mathfrak{w})$ is a Cartan subalgebra of $\mathfrak{g}$, which we shall identify with a subspace of $\mathfrak{g}'$ via (57). Let $\mathsf{V}'' = \{v \in \mathsf{V}';\ xv = 0 \text{ for all } x \in \mathfrak{h}\}$. Then the restriction of the form $(\ ,\ )'$ to V'' is non-degenerate, so that $\mathsf{V}' = \mathsf{V}''^{\perp} \oplus \mathsf{V}''$ and, in terms of this decomposition, the centralizer of $\mathfrak{h}$ in $\mathfrak{g}$ is equal to

$$\mathfrak{z}' = \mathfrak{h} \oplus \mathfrak{z}'', \tag{58}$$

where $\mathfrak{z}''$ is the Lie algebra of the isometry group Z'' of $(\mathsf{V}'',\ (\ ,\ )')$. Let Z' be the normalizer of $\mathfrak{z}'$ in G'. Then, according to (58),

$$\mathcal{U}(\mathfrak{z}')^{\mathrm{Z}'} = \mathcal{U}(\mathfrak{h})^{W} \otimes \mathcal{U}(\mathfrak{z}'')^{\mathrm{Z}''}, \tag{59}$$

where W is the Weyl group as in (52). Fix any Cartan subalgebra $\mathfrak{h}'' \subseteq \mathfrak{z}''$. Then $\mathfrak{h}' = \mathfrak{h} \oplus \mathfrak{h}''$ is a Cartan subalgebra of $\mathfrak{z}'$ and of $\mathfrak{g}'$.

Let $\epsilon_{\mathfrak{z}''} : \mathcal{U}(\mathfrak{z}'') \to \mathbb{C}$ be the algebra homomorphism by which the algebra of the differential operators, $\mathcal{U}(\mathfrak{z}'')$, acts on the constant functions on Z''. In these terms we have the following algebra homomorphism

$$\mathcal{C} : \mathcal{U}(\mathfrak{g}')^{\mathrm{G}'} \underset{\gamma_{\mathfrak{z}'/\mathfrak{h}'}^{-1}\circ\gamma_{\mathfrak{g}'/\mathfrak{h}'}}{\to} \mathcal{U}(\mathfrak{z}')^{\mathrm{Z}'} \underset{1\otimes\epsilon_{\mathfrak{z}''}}{\to} \mathcal{U}(\mathfrak{h})^{W} \underset{\gamma_{\mathfrak{g}/\mathfrak{h}}^{-1}}{\to} \mathcal{U}(\mathfrak{g})^{\mathrm{G}}. \tag{60}$$

The point is that for any two representations $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$, $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$ in Howe's correspondence, as in (50), with the infinitesimal characters γ_{Π}, $\gamma_{\Pi'}$ respectively,

$$\gamma_{\Pi'} = \gamma_{\Pi} \circ \mathcal{C}, \tag{61}$$

see [Prz04, Theorem 5.6]. The relation (61) was computed in slightly different terms in [Prz96]. This was part of the second author's PhD research under the guidance of Roger Howe. The construction presented in [Prz04] is based on the moment maps (54) and Harish-Chandra's radial component map (53).

8. Characters

Let G be a real reductive group with a maximal compact subgroup K. Every irreducible admissible $(\mathfrak{g}, \mathrm{K})$-module may be realized as the Harish-Chandra module V^o of a representation Π of G on a Hilbert space V. This representation does not have to be unitary. Harish-Chandra proved, [Har51b, Theorem 2], that for any test function $f \in \mathbb{C}^\infty(\mathrm{G})$, the operator

$$\Pi(f) = \int_{\mathrm{G}} f(g)\Pi(g)\, dg \tag{62}$$

is of trace class and that the formula

$$\Theta_\Pi(f) = tr\, \Pi(f) \qquad (f \in C_c^\infty(\mathrm{G})) \tag{63}$$

defines a distribution Θ_Π on G. This is the distribution character of the representation Π.

Let us identify $T^*\mathrm{G} = \mathrm{G} \times \mathfrak{g}^*$ by

$$df(g) = \frac{d}{dt} f(g\, exp(tx))|_{t=0} \qquad (f \in C_c^\infty(\mathrm{G}),\ g \in \mathrm{G},\ x \in \mathfrak{g}). \tag{64}$$

This way the fiber $\{g\} \times \mathfrak{g}^*$ of $T^*\mathrm{G}$, over any point of $g \in \mathrm{G}$ may be viewed as $\mathfrak{g}^*$. In these terms the wave front set of the representation Π, $WF(\Pi) \subseteq \mathfrak{g}^*$, is defined as the fiber of $WF(\Theta_\Pi)$ over the identity.

Rossmann proved, [Ros95, Theorem 3.4], that this is the largest fiber and even more that it is equal to the closure of the union of the fibers of the wave front sets of all the distributions

$$tr(\Pi(f)T) \qquad (f \in C_c^\infty(\mathrm{G})), \tag{65}$$

where T varies through the space of all the trace class operators on V. This last set was defined as the wave front set of the representation in [How81].

Since an irreducible representation is quasisimple, the character Θ_Π satisfies the following system of differential equations:

$$z\,\Theta_\Pi = \gamma_\Pi(z)\,\Theta_\Pi \qquad (z \in \mathcal{U}(\mathfrak{g})^{\mathrm{G}}) \tag{66}$$

We see from the definition of the wave front set (15) and from (66) that $WF(\Theta_\Pi)$ is contained in the intersection of the sets of the characteristic points of the differential operators

$$z - \gamma_\Pi(z) \qquad (z \in \mathcal{U}(\mathfrak{g})^{\mathrm{G}}). \tag{67}$$

In particular, $WF(\Pi)$ is contained in the intersection of the zero sets of the non-constant G-invariant polynomials on $\mathfrak{g}^*$, which is equal to the nilpotent cone in $\mathfrak{g}^*$, [Wal88, sec. 8.A.4.2].

Under an appropriate choice of the Killing forms on $\mathfrak{g}$ and $\mathfrak{g}'$ we may identify the unnormalized maps (54) with

$$\begin{aligned} &\tau : \mathsf{W} \to \mathfrak{g}^*,\ \tau(w)(x) = \langle xw, w\rangle \\ &\tau' : \mathsf{W} \to \mathfrak{g}'^*,\ \tau'(w)(x') = \langle x'w, w\rangle \qquad (x \in \mathfrak{g},\ x' \in \mathfrak{g}',\ w \in \mathsf{W}). \end{aligned} \tag{68}$$

With some effort one can see from the formula (30) that these maps occur naturally in the restriction of the oscillator representation to the dual pair. Hence, as shown in [Prz93, Corollary 2.8],

$$WF(\Pi) \subseteq \tau(W) \qquad (\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)). \tag{69}$$

Let $I_\Pi \subseteq \mathcal{U}(\mathfrak{g})$ denote the annihilator of the Harish-Chandra module of the representation Π. If we think of the universal enveloping algebra as the algebra of the left invariant differential operators on the group, then I_Π consists of the operators which annihilate the character Θ_Π. In particular the intersection of the characteristic sets of all these operators contains the wave front set of the character. Let us view the principal symbol of any such operator as a function on the complexified cotangent bundle $\mathrm{G} \times \mathfrak{g}^*_{\mathbb{C}}$. Then the joint zero set of the principal symbols is of the form $\mathrm{G} \times \mathcal{V}(I_\Pi)$. The set $\mathcal{V}(I_\Pi) \subseteq \mathfrak{g}^*_{\mathbb{C}}$ is called the associated variety of the annihilator of the Harish-Chandra module of Π. As we just noticed, $WF(\Pi) \subseteq \mathcal{V}(I_\Pi)$ is a much rougher invariant of the character. In particular, it is relatively easy to show that for any $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$, $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}', \omega)$ in Howe's correspondence,

$$\mathcal{V}(I_{\Pi'}) \subseteq \tau'\tau^{-1}(\mathcal{V}(I_\Pi)), \tag{70}$$

where the moment maps (68) are extended to the complexifications of W, $\mathfrak{g}^*$ and $\mathfrak{g}'^*$, see [Prz91, Theorem 7.1].

The wave front set of the character Θ of the oscillator representation, (30), is given by

$$WF(\Theta) = \{(g, \xi) \in \widetilde{\mathrm{Sp}} \times \mathfrak{sp}^*;\ \xi \in WF_1(\Theta),\ Ad(g)^*(\xi) = \xi\}, \tag{71}$$

where the fiber over the identity, $WF_1(\Theta) = \mathrm{O}_{min}$ is one of the two minimal non-zero nilpotent coadjoint orbits in $\mathfrak{sp}^*$, [Prz00, Lemma 12.2]. (It would be interesting to know for which characters does the formula (71) hold, without any specific description of the fiber over the identity.) The formula is a key to a construction of an operator from the space of the invariant eigen-distributions on $\widetilde{\mathrm{G}}$ to the space of the invariant eigen-distributions on $\widetilde{\mathrm{G}}'$, [Prz00], [BP06]. (We keep working under the assumption that the rank of G is less or equal to the rank of G′.) We recall it below.

A maximal compact subgroup $\mathrm{K} \subseteq \mathrm{G}$ is determined by a Cartan involution $\theta : \mathrm{G} \to \mathrm{G}$. (For example, if $\mathrm{G} = \mathrm{GL}_n(\mathbb{R})$ and $K = \mathrm{O}_n$ then $\theta(g) = (g^t)^{-1}$.) Specifically, K consists of the θ-fixed points in G. Let $\mathrm{P} \subseteq \mathrm{G}$ be the subset of the elements $g \in \mathrm{G}$ such that $\theta(g) = g^{-1}$. Then G = KP.

Any Cartan subgroup $\mathrm{H} \subseteq \mathrm{G}$ is conjugate to one which is invariant under θ. Thus let H be a θ-stable Cartan subgroup of G. Set $\mathrm{A} = \mathrm{H} \cap \mathrm{P}$. This is called the vector part of H, [Wal88].

Denote by $\mathrm{A}' \subseteq \mathrm{Sp}$ the centralizer of A and let $\mathrm{A}'' \subseteq \mathrm{Sp}$ be the centralizer of A'. There is a measure $d\dot{w}$ on the quotient space $\mathrm{A}''\backslash \mathsf{W}$ defined by

$$\int_{\mathsf{W}} \phi(w)\, dw = \int_{\mathrm{A}''\backslash \mathsf{W}} \int_{\mathrm{A}''} \phi(aw)\, da\, d\dot{w}. \tag{72}$$

Let $\widetilde{\mathrm{A}}'$ be the preimage of A' in the metaplectic group. Recall the embedding $T : \widetilde{\mathrm{Sp}} \to S^*(\mathsf{W})$ defined in Theorem 1. The formula

$$Chc(f) = \int_{\mathrm{A}''\backslash \mathsf{W}} \int_{\widetilde{\mathrm{A}'}} f(g)T(g)(w)\, dg\, d\dot{w} \qquad (f \in C_c^\infty(\widetilde{\mathrm{A}}')), \tag{73}$$

where each consecutive integral is absolutely convergent, defines a distribution on $\widetilde{\mathrm{A}}'$, [Prz00, Lemma 2.9]. Fix a regular element $h \in \mathrm{H}^{reg}$. Let $\widetilde{h}$ be an element in the preimage of h in the metaplectic group. The intersection of the wave front set of the distribution (73) with the conormal bundle of the embedding

$$\widetilde{\mathrm{G}'} \ni \widetilde{g} \to \widetilde{h}\widetilde{g'} \in \widetilde{\mathrm{A}''} \tag{74}$$

is empty (i.e. contained in the zero section), [Prz00, Proposition 2.10]. Hence there is a unique restriction of the distribution (73) to $\widetilde{\mathrm{G}}$, denoted $Chc_{\widetilde{h}}$.

Harish-Chandra's Regularity Theorem, [Har63, Theorem 2], implies that the character of an irreducible representation coincides with a function multiplied by the Haar measure. Thus for $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}})$ we may consider the following integral

$$\int_{\widetilde{\mathrm{H}^{reg}}} \Theta_\Pi(\widetilde{h}^{-1}) |det(Ad(h^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}}| Chc_{\widetilde{h}}(f)\, d\widetilde{h} \qquad (f \in C_c^\infty(\widetilde{\mathrm{G}'}). \tag{75}$$

In fact, this integral is absolutely convergent, [Prz00, Theorem 2.14].

Recall the Weyl - Harish-Chandra integration formula

$$\int_{\widetilde{\mathrm{G}}} f(g)\, dg = \sum \frac{1}{|\mathcal{W}(\mathrm{H},\mathrm{G})|} \int_{\widetilde{\mathrm{H}^{reg}}} |det(Ad(h^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}}| \int_{\widetilde{\mathrm{G}}/\widetilde{\mathrm{H}}} f(g\widetilde{h}g^{-1})\, d\dot{g}\, d\widetilde{h}, \tag{76}$$

where $\mathcal{W}(\mathrm{H},\mathrm{G})$ is the Weyl group of H in G and the summation is over a maximal family of mutually non-conjugate (θ-stable) Cartan Subgroups $\widetilde{\mathrm{G}}$. In terms of (76), set

$$\Theta'_\Pi(f) = C_\Pi \sum \frac{1}{|\mathcal{W}(\mathrm{H},\mathrm{G})|} \int_{\widetilde{\mathrm{H}^{reg}}} \Theta_\Pi(\widetilde{h}^{-1}) |det(Ad(h^{-1} - 1)_{\mathfrak{g}/\mathfrak{h}}| Chc_{\widetilde{h}}(f)\, d\widetilde{h}, \tag{77}$$

where C_Π is a constant. This is an invariant distribution on $\widetilde{\mathrm{G}'}$. In fact, with the appropriate normalization of all the measures involved, [BP06, Theorem 4], Θ'_Π is an invariant eigen-distribution whose infinitesimal character is related to the infinitesimal character of Θ_Π by (61). There are reasons to believe that (for an appropriate constant C_Π) Θ'_Π coincides with the character of the representation Π'_1, (49). Since quite often, $\Pi'_1 = \Pi'$, the above construction could explain Howe's correspondence on the level of characters. However there is no proof of the equality $\Theta'_\Pi = \Theta_{\Pi'_1}$, in general yet. For a precise conjecture see [BP06].

9. Unitary representations

If a unitary representation $\Pi \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$ occurs in the Hilbert space $\mathrm{L}^2(X)$ then so does the corresponding representation $\Pi' \in \mathcal{R}(\widetilde{\mathrm{G}}, \omega)$. In particular, Π' is also unitary. This is interesting if one has a different construction of Π' from which the unitarity is not clear. This idea was studied by J. Adams in [Ada83] for the pair $\mathrm{O}_{p,q}$, $\mathrm{Sp}_{2n}(\mathbb{R})$.

Conversely, for some pairs one may describe Howe's correspondence completely in terms of Langlands' parameters, identify the unitary duals of the group involved and see how does the correspondence relate to unitarity. For the pair $\mathrm{O}_{2,2}$, $\mathrm{Sp}_4(\mathbb{R})$ this was done in [Prz89], with the conclusion that the correspondence maps the unitary representations of the orthogonal group to the unitary representations of the symplectic group.

One says that a dual pair G, G′ of isometry groups is in the stable range if the dimension of the defining module for G is less then or equal to the dimension of the maximal isotropic subspace of the defining module for G′. (For example the pair $\mathrm{O}_{p,q}$, $\mathrm{Sp}_{2n}(\mathbb{R})$ is in the stable range if $p + q \leq n$.) Jian-Shu Li has shown that Howe's correspondence maps a unitary representation of $\widetilde{\mathrm{G}}$ to a unitary representation Π' of $\widetilde{\mathrm{G}'}$ (without the assumption that these representations occur in the Hilbert space $\mathrm{L}^2(X)$), [Li89]. This theorem was generalized in [Prz93] and [He03], and none of these works appeals to Langlands classification.

For the latest along these lines see [ABP+07, Theorem 1.5].

References

[AB95] J. Adams and D. Barbasch. Reductive dual pairs correspondence for complex groups. *J. Funct. Anal.*, 132:1–42, 1995.

[ABP+07] J. Adams, D. Barbasch, A. Paul, P. Trapa, and D. A. Vogan. Unitary Shimura corresondence for split real groups. *Journal of the AMS*, 20:701–751, 2007.

[Ada83] J. Adams. Discrete spectrum of the dual pair $(O(p,q), Sp(2m,\mathbb{R})$. *Invent. Math*, 74:449–475, 1983.

[Bar89] D. Barbasch. The unitary dual for complex classical Lie groups. *Invent. Math.*, 96:103–176, 1989.

[BP06] F. Bernon and T. Przebinda. The Cauchy Harish-Chandra integral and the invariant eigendistributions. *Preprint*, 2006.

[EHW83] T. J. Enright, R. Howe, and N. R. Wallach. A classification of unitary highest weight modules. *Proceedings of Utah Conference, 1982*, pages 97–143, 1983.

[Glo82] P. Glowacki. A Calculus of Symbols and Convolution Semigroups on the Heisenberg Group. *Studia Math.*, 72:291–321, 1982.

[Har51a] Harish-Chandra. Representations of Semisimple Lie Groups II. *Proceedings of the National Academy of Sciences*, 37:362–365, 1951.

[Har51b] Harish-Chandra. Representations of Semisimple Lie Groups III. Characters. *Proceedings of the National Academy of Sciences*, 37:366–369, 1951.

[Har53] Harish-Chandra. Representations of Semisimple Lie Groups on a Banach space I. *Trans. Amer. Math. Soc*, 75:185–243, 1953.

[Har56] Harish-Chandra. The Characters of Semisimple Lie Groups. *Trans. Amer. Math. Soc*, 83:98–163, 1956.

[Har63] Harish-Chandra. Invariant eigendistributions on semisimple Lie groups. *Bull. Amer. Math. Soc.*, 69:117–123, 1963.

[He03] H. He. Unitary Representations and Theta Correspondence for Type I Classical Groups. *J. Funct. Anal.*, 199:92–121, 2003.

[Hör83] L. Hörmander. *The Analysis of Linear Partial Differential Operators I.* Springer Verlag, 1983.

[Hör85] L. Hörmander. *The Analysis of Linear Partial Differential Operators III.* Springer Verlag, 1985.

[How79] R. Howe. θ-series and invariant theory. In *Automorphic forms, representations and L-functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Part 1*, Proc. Sympos. Pure Math., XXXIII, pages 275–285. Amer. Math. Soc., Providence, R.I., 1979.

[How80] R. Howe. Quantum mechanics and partial differential equations. *J. Funct. Anal.*, 38:188–254, 1980.

[How81] R. Howe. Wave Front Sets of Representations of Lie Groups. In *Automorphic forms, Representation Theory and Arithmetic*, pages 117–140. Tata Institute of Fundamental Research, Bombay, 1981.

[How85] Howe, R. Dual pairs in physics, harmonic oscillators, photons, electrons and singletons. volume 21 of *Lecture notes in Appl. Math*, pages 179–207. Amer. Math. Soc., Providence, RI, 1985.

[How88] R. Howe. The oscillator semigroup. *Proc. Symp. Pure Math., Amer. Math. Soc.*, 48:61–132, 1988.

[How89a] R. Howe. Remarks on Classical Invariant Theory. *Trans. Amer. Math. Soc.*, 313:539–570, 1989.

[How89b] R. Howe. Transcending Classical Invariant Theory. *J. Amer. Math. Soc. 2*, 2:535–552, 1989.

[HR63] E. Hewitt and K. A. Ross. *Abstract Harmonic Analysis.* Springer Verlag, 1963.

[Kir62] A. A. Kirillov. Unitary representations of nilpotent Lie groups. *Uspehi Mat. Nauk*, 17(4 (106)):57–110, 1962.

[KV78] M. Kashiwara and M. Vergne. On the segal-shale-weil representation and harmonic polynomials. *Invent. Math.*, 44:1–47, 1978.

[Li89] Jian-Shu Li. Singular unitary representations of classical groups. *Invent. Math.*, 97(2):237–255, 1989.

[Pau98] A. Paul. Howe correpondence for real unitary groups I. *J. Funct. Anal.*, 159:384–431, 1998.

[Pau00] A. Paul. Howe correspondence for real unitary groups II. *Proc. Amer. Math. Soc.*, 128:3129–3136, 2000.

[Prz89] T. Przebinda. The duality correspondence for the pair $(O(2,2), \mathrm{Sp}(4,\mathbb{R}))$. *Memoirs of the AMS*, 403, 1989.

[Prz91] T. Przebinda. Characters, dual pairs, and unipotent representations. *J. Funct. Anal.*, 98(1):59–96, 1991.

[Prz93] T. Przebinda. Characters, dual pairs, and unitary representations. *Duke Math. J. 69 (1993), no. 3, 547-592*, 69(3):547–592, 1993.

[Prz96] T. Przebinda. The duality correspondence of infinitesimal charcters. *Coll. Math*, 70:93–102, 1996.

[Prz00] T. Przebinda. A Cauchy Harish-Chandra Integral, for a real reductive dual pair. *Inven. Math.*, 141(2):299–363, 2000.

[Prz04] T. Przebinda. A Capelli Harish-Chandra Homomorphism. *Trans.Amer.Math.Soc.*, 356(3):1121–1154, 2004.

[Prz06] T. Przebinda. Local Geometry of Orbits for an Ordinary Classical Lie Supergroup. *Central European Journal of Mathematics*, 4:449–506, 2006.

[RH92] E. Tan R. Howe. *Nonabelian harmonic analysis. Applications of* $\mathrm{SL}(2, R)$. Springer Verlag, 1992.

[Ros95] W. Rossmann. Picard-lefschetz theory and characters of semisimple a lie group. *Invent. Math.*, 121:579–611, 1995.

[Rud91] Rudin, W. *Functional Analysis.* McGraw-Hill, Inc, 1991.

[SW71] E.M. Stein and G. Weiss. *Introduction to Fourier Analysis on Euclidean Spaces.* Princeton University Press, Princeton, NJ, 1971.

[Vog86] D. Vogan. The unitary dual of $\mathrm{GL}(n)$ over an archimedean field. *Invent. Math.*, 83:449–505, 1986.

[Vog94] D. Vogan. The unitary dual of G_2. *Invent. Math.*, 116:677–791, 1994.

[Wal88] N. Wallach. *Real Reductive Groups I.* Academic Press, 1988.

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