

THE CAUCHY HARISH-CHANDRA INTEGRAL AND THE INVARIANT EIGENDISTRIBUTIONS

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ABSTRACT. In this paper, we prove that the Cauchy Harish-Chandra integral maps invariant eigendistributions to invariant eigendistributions with the correct infinitesimal character, and that it maps the orbital integrals on the larger group to the orbital integrals on the smaller one. This is the last paper of a series of three.

THE MAIN RESULTS

Let W be a finite dimensional vector space over the reals, with a non-degenerate symplectic form $\langle \cdot, \cdot \rangle$. Let \mathcal{J} be a positive compatible complex structure on W , $\mathrm{Sp}(W)$ (resp. $\mathfrak{sp}(W)$) the symplectic group (resp. the symplectic Lie algebra) associated to $\langle \cdot, \cdot \rangle$ and let $\chi(r) = \exp(2\pi ir)$, $r \in \mathbb{R}$. Fix a Lebesgue measure dw on W so that

$$\int_W \chi\left(\frac{i}{2}\langle \mathcal{J}w, w \rangle\right) dw = 1.$$

The conjugation by \mathcal{J} is a Cartan involution θ on $\mathfrak{sp}(W)$. The formula

$$\tilde{\kappa}(x, y) = -\mathrm{tr}(\theta x, y) \quad (x, y \in \mathfrak{sp}(W))$$

defines a positive definite symmetric form on $\mathfrak{sp}(W)$. We normalize the Lebesgue measure μ on any subspace of $\mathfrak{sp}(W)$ by requiring that the volume of the unit cube is 1. For any Lie subgroups $F \subseteq E \subseteq \mathrm{Sp}(W)$, the measure μ induces the left invariant Haar measure on E and invariant measures on the quotients E/F and $F \backslash E$. We shall denote these induced measures also by μ .

Let (G, G') be a reductive dual pair of $\mathrm{Sp}(W)$ (see [How89] for the definition), with the rank of G' less or equal to the rank of G . Denote by $\mathfrak{g}, \mathfrak{g}'$ the Lie algebras of G, G' respectively. We lift the Cartan involution θ to the group and assume (as we may) that G and G' are preserved by θ . Let H' be a Cartan subgroup of G' preserved by θ . Consider the Cartan decomposition of H' : $H' = T'A'$ where T' (resp. A') is the compact (resp. split) part of H' . Consider the commutant A'' (resp. A''') of A' (resp. A'') in $\mathrm{Sp}(W)$. Then (A'', A''') is a reductive dual pair of $\mathrm{Sp}(W)$, see [Prz00]. Let $V'_c = \{v \in V' \mid av = v \ \forall a \in A'\}$. There exists a unique complement V'_s of V'_c in V' such that the decomposition

$$V' = V'_c \oplus V'_s$$

is preserved by H' . As $A''' \subset H'$, we may consider

$$A_s''' = \{v \in A''' \mid v|_{V'_c} = \mathrm{id}\}.$$

Then $A_s''' = A_s'''$ if and only if $V'_c = 0$ and $A_s''' = A_s''' \times \{\pm \mathrm{id}|_{V'_c}\}$ otherwise. There exists a dense open subset $W_{A_s'''}'$ of W such that the quotient $A_s''' \backslash W_{A_s'''}'$ is a smooth manifold. Define the measure μ on the quotient $A_s''' \backslash W_{A_s'''}'$ by

$$\int_W f(w) dw = \int_{A_s''' \backslash W_{A_s'''}'} \int_{A_s'''} f(aw) d\mu(a) d\mu(A_s''' w).$$

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Let

$$\chi_x(w) = \chi\left(\frac{1}{4}\langle xw, w \rangle\right), \quad (x \in \mathfrak{sp}(W), w \in W).$$

Recall [Prz00, p. 302] the Cauchy Harish-Chandra integral on the Lie algebra:

$$\widetilde{chc}(x' + x) = \int_{A_s''' \setminus W_{A_s'''}} \chi_{x'+x}(w) d\mu(A_s''' w) \quad (x' \in \mathfrak{h}'^{reg}, x \in \mathfrak{g}).$$

Define the normalized Cauchy Harish-Chandra integral by

$$chc(x' + x) = \frac{1}{\mu(A_s''' \setminus H')} \widetilde{chc}(x' + x) \quad (x' \in \mathfrak{h}'^{reg}, x \in \mathfrak{g}).$$

Let $\widetilde{\mathrm{Sp}}(W)$ be the connected two fold cover of $\mathrm{Sp}(W)$. This is the metaplectic group with the canonical surjection

$$\widetilde{\mathrm{Sp}}(W) \longrightarrow \mathrm{Sp}(W). \quad (1)$$

For a subset E of $\mathrm{Sp}(W)$, denote by \widetilde{E} the preimage of E in $\widetilde{\mathrm{Sp}}(W)$.

Let $S'(W)$ be the space of temperate distributions on W . Recall [Prz00, Theorem 2.8] Howe's embedding of the metaplectic group into the space of temperate distributions on the symplectic space:

$$T : \widetilde{\mathrm{Sp}}(W) \longrightarrow S'(W).$$

Recall the definition of the Cauchy Harish-Chandra integral on the group \widetilde{G}' , [Prz00, Definition 2.11]:

$$\widetilde{Chc}(x'x) = \int_{A_s''' \setminus W_{A_s'''}} T(x'x)(w) d\mu(A_s''' w) \quad (x' \in \widetilde{H}'^{reg}, x \in \widetilde{G}).$$

Define the normalized integral by

$$Chc(x'x) = \frac{1}{\mu(A_s''' \setminus H')} \widetilde{Chc}(x'x) \quad (x' \in \widetilde{H}'^{reg}, x \in \widetilde{G}).$$

Statement on the Lie algebra level. For $\phi \in \mathcal{D}(\mathfrak{g})$ (the space of the compactly supported smooth functions on \mathfrak{g}) let

$$chc(\phi)(x') = \int_{\mathfrak{g}} chc(x' + x)\phi(x) d\mu(x) \quad (x' \in \mathfrak{h}'^{reg}).$$

The above formula defines a G' -invariant function, $chc(\phi)$, on \mathfrak{g}'^{reg} . Let us denote by $\widetilde{\mathcal{I}}(\mathfrak{g}')$ the space of unnormalized orbital integrals on \mathfrak{g}' with no condition on the support. This means that $\widetilde{\mathcal{I}}(\mathfrak{g}')$ is the set of all the functions $\psi \in C^\infty(\mathfrak{g}'^{reg})$ such that the functions

$$|\det(\mathrm{ad}(x))_{\mathfrak{g}'/\mathfrak{g}'^x}|^{1/2}\psi(x) \quad (x \in \mathfrak{g}'^{reg})$$

satisfies the conditions I_1, I_2 and I_3 of [Bou94a, §3.2] and we don't assume that the condition I_4 is satisfied. We denote also by $\mathcal{I}(\mathfrak{g})$ the space of unnormalized orbital integrals on \mathfrak{g} assuming this time that all the conditions I_1, I_2, I_3 and I_4 are satisfied. In other words, by a result of Bouaziz [Bou94a], $\mathcal{I}(\mathfrak{g})$ is the space of all the functions of the form

$$\int_{G/G^x} \psi(g.x) d\mu(gG^x) \quad (x \in \mathfrak{g}^{reg}, \psi \in \mathcal{D}(\mathfrak{g})).$$

The first result of this paper is :

Theorem 1. *Let $\phi \in \mathcal{D}(\mathfrak{g})$ then $chc(\phi) \in \widetilde{\mathcal{I}}(\mathfrak{g}')$.*

Concerning the support, the functions in the range of chc do not necessarily satisfied the condition I_4 . The functions restricted to a given Cartan subalgebra \mathfrak{h}' have a compact support modulo the elliptic part of \mathfrak{h}' .

Statements on the Lie group level. Let G_1 denote the Zariski identity component of G multiplied by the center of G . Then $G_1 = G$ unless G is a real, even orthogonal group $O_{2p,2q}$. Similarly we define G'_1 . Let $\phi \in \mathcal{D}(\tilde{G}_1)$ (the space of the compactly supported smooth functions on \tilde{G}_1). Define $Chc(\phi)$ to be a \tilde{G}' -invariant function on \tilde{G}_1^{reg} by the formula

$$Chc(\phi)(x') = \int_{\tilde{G}_1} Chc(x'x)\phi(x) d\mu(x) \quad (x' \in \tilde{H}^{reg}).$$

Let us denote by $\mathcal{I}(\tilde{G}_1)$ the space of unnormalized orbital integrals on \tilde{G}_1 . This means that $\mathcal{I}(\tilde{G}_1)$ is the set of function of the form

$$\tilde{G}_1^{reg} \ni x \longrightarrow \int_{\tilde{G}_1/\tilde{G}_1^x} \psi(g.x)d\mu(g\tilde{G}_1^x) \in \mathbb{C},$$

where $\psi \in \mathcal{D}(\tilde{G}_1)$, see [Bou94b]. We denote by $I_{\tilde{G}}$ the map from $\mathcal{D}(\tilde{G}_1)$ onto $\mathcal{I}(\tilde{G}_1)$, defined above, and similarly for \tilde{G}' . One of the main result of this paper is

Theorem 2. *Let $\phi \in \mathcal{D}(\tilde{G}_1)$. Then $Chc(\phi) \in \mathcal{I}(\tilde{G}'_1)$. Moreover, the resulting map*

$$Chc : \mathcal{D}(\tilde{G}_1) \rightarrow \mathcal{I}(\tilde{G}'_1)$$

is continuous and, if $I_{\tilde{G}}(\phi)$ is 0, then $Chc(\phi)$ is also 0.

This theorem proves that Chc induces a map

$$Chc : \mathcal{I}(\tilde{G}_1) \rightarrow \mathcal{I}(\tilde{G}'_1). \quad (2)$$

Concerning the support of the functions in the range of Chc , their restrictions to a given Cartan subgroup \tilde{H}' of \tilde{G}' have a compact support modulo the compact part of \tilde{H}' . Therefore, they are compactly supported. Let $\mathcal{D}'(\tilde{G}'_1)$ be the space of distributions on \tilde{G}'_1 and let $\mathcal{D}'^{\tilde{G}'}(\tilde{G}'_1)$ be the space of \tilde{G}' -invariant distributions on \tilde{G}'_1 . Recall, [Bou94a] that the transpose of the map $I_{\tilde{G}}$, induces an isomorphism between $\mathcal{I}(\tilde{G}'_1)'$ (the dual of $\mathcal{I}(\tilde{G}'_1)$) and $\mathcal{D}'^{\tilde{G}'}(\tilde{G}'_1)$. Hence, we have

$$\begin{aligned} {}^t I_{\tilde{G}'} : \mathcal{I}(\tilde{G}'_1)' &\xrightarrow{\cong} \mathcal{D}'^{\tilde{G}'}(\tilde{G}'_1), \\ {}^t I_{\tilde{G}} : \mathcal{I}(\tilde{G}_1)' &\xrightarrow{\cong} \mathcal{D}'^{\tilde{G}}(\tilde{G}_1). \end{aligned} \quad (3)$$

We denote also by Chc the resulting map

$$Chc : \mathcal{D}'^{\tilde{G}'}(\tilde{G}'_1) \longmapsto \mathcal{D}'^{\tilde{G}}(\tilde{G}_1) \quad (4)$$

induced by the transpose of Chc and the isomorphisms (3). Thus for $u' \in \mathcal{D}'^{\tilde{G}'}(\tilde{G}'_1)$,

$$Chc(u') = ({}^t I_{\tilde{G}'})^{-1} (u') \circ Chc. \quad (5)$$

In particular, if u' is given in terms of a locally integrable function, then, for $\phi \in \mathcal{D}(\tilde{G}_1)$,

$$Chc(u')(\phi) = \sum \frac{1}{|W(\tilde{H}')|} \int_{\tilde{H}^{reg}} |\det(1 - \text{Ad}(h^{-1}))_{\mathfrak{g}'/\mathfrak{b}'}| u'(h) Chc(\phi)(h) d\mu(h), \quad (6)$$

where the summation is over a maximal family of mutually non-conjugate Cartan subgroups $H' \subseteq \tilde{G}'$ and $|W(H')|$ stands for the cardinality of the Weyl group of H' in \tilde{G}' .

We denote by $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ (resp. $\mathcal{U}(\mathfrak{g}'_{\mathbb{C}})$) the enveloping algebra of $\mathfrak{g}_{\mathbb{C}}$ (resp. $\mathfrak{g}'_{\mathbb{C}}$). Consider the Capelli Harish-Chandra homomorphism (cf. equality (61)) :

$$\mathcal{C}_{\mathfrak{g},\mathfrak{g}'} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G \longrightarrow \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}.$$

Let L be the left regular representation of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ on $\mathcal{D}(\tilde{G})$ (cf. equality (55)) and similarly for \mathfrak{g}' .

Theorem 3. *Let $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$. Then for $\phi \in \mathcal{D}(\tilde{G}_1)$*

$$Chc(L(\tilde{z})\phi) = L(\mathcal{C}_{\mathfrak{g},\mathfrak{g}'}(z))Chc(\phi),$$

where $z \rightarrow \tilde{z}$ be the involution on the universal enveloping algebra, extending the map $\mathfrak{g}_{\mathbb{C}} \ni z \rightarrow -z \in \mathfrak{g}_{\mathbb{C}}$.

The action of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ on $\mathcal{D}(\tilde{G})$ induces an action on $\mathcal{D}'(\tilde{G})$ denoted also by L . By definition, we have

$$L(z)u(\phi) = u(L(\tilde{z})\phi)$$

for $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$, $u \in \mathcal{D}'(\tilde{G})$ and $\phi \in \mathcal{D}(\tilde{G})$. Moreover, $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ stabilizes $\mathcal{D}'^{\tilde{G}}(\tilde{G})$. The same holds for \tilde{G}' . The following theorem explains the title of the paper.

Theorem 4. *Let $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$ and $u' \in \mathcal{D}'^{\tilde{G}'}(\tilde{G}'_1)$. Then*

$$Chc(L(\mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(z))u') = L(\tilde{z})Chc(u').$$

This result was the main aim of our project. One might deduce it directly from Theorem 7.4 in [BP06a], however it is conceptually easier to see that it follows from Theorems 2 and 3, because the action of the centers of the universal enveloping algebras intertwine the maps (3).

Relation with Howe's correspondence. Let $\mathfrak{d} \in \widetilde{\text{Sp}}(W)$ be any element in the preimage of $-1 \in \text{Sp}(W)$. Let Π' be an irreducible admissible representation of \tilde{G}' and let $\Theta_{\Pi'}$ denote the distribution character of Π' . Denote by $\chi_{\Pi'}(\mathfrak{d})$ the scalar by which $\Pi'(\mathfrak{d})$ acts on the Hilbert space of Π' . Theorem 2 implies that for a test function $\phi \in \mathcal{D}(\tilde{G})$ there is a test function $\phi' \in \mathcal{D}(\tilde{G}'_1)$ whose orbital integrals coincide with $Chc(\phi)$. Let

$$\Theta'_{\Pi'}(\phi) = \chi_{\Pi'}(\mathfrak{d})\Theta(\mathfrak{d}) \int_{\tilde{G}'_1} \Theta_{\Pi'}(g^{-1})\phi'(g) d\mu(g). \quad (7)$$

Though the function ϕ' is not uniquely determined, its orbital integrals are. Hence the formula (7) defines an invariant distribution on \tilde{G}'_1 . (Recall that $\Theta_{\Pi'}(g^{-1}) = \bar{\Theta}_{\Pi'}(g)$ if Π' is unitary.)

Let \mathcal{H}^∞ denote the space of the smooth vectors in the Hilbert space of the oscillator representation ω corresponding to the character χ of the additive group of the real numbers, as in [How88]. Suppose the representation Π' occurs as a quotient of \mathcal{H}^∞ by a closed invariant subspace \mathcal{H}_1^∞ . Let $\mathcal{H}_{\Pi'}^\infty$ be the intersection of all such subspaces \mathcal{H}_1^∞ . As shown in [How89],

$$\mathcal{H}^\infty / \mathcal{H}_{\Pi'}^\infty \text{ is isomorphic to } \Pi'_1 \otimes \Pi'$$

as a $\tilde{G} \times \tilde{G}'$ -module, where Π'_1 is a finitely generated admissible quasisimple representation of \tilde{G} , which has a unique quotient Π (the "theta lift" of Π').

Here we offer a more precise version of a conjecture formulated in [Prz00, Conjecture 2.18].

Conjecture. *Suppose Π' occurs as a quotient of \mathcal{H}^∞ and the function $\Theta_{\Pi'}$ is supported in \tilde{G}'_1 , i.e. the restriction to the complement is zero: $\Theta_{\Pi'}|_{\tilde{G}' \setminus \tilde{G}'_1} = 0$. Then*

$$\Theta'_{\Pi'} = \Theta_{\Pi'_1}|_{\tilde{G}'_1}.$$

It is clear from Theorem 4 that if $\gamma' : \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'} \rightarrow \mathbb{C}$ is the infinitesimal character of Π' then $\Theta'_{\Pi'}$ is an $\gamma' \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}$ -eigendistribution, as are $\Theta_{\Pi'_1}$ and Θ_{Π} . Thus $\Theta'_{\Pi'}$ is an invariant eigendistribution with the correct infinitesimal character!

If our dual pair is in the "deep stable range" with G' the smaller member (see [DP96]), and Π' is genuine and unitary, then $\Pi'_1 = \Pi$ and $\Theta'_{\Pi'} = \Theta_{\Pi}$. (In fact [DP96] was the main motivation for [Prz00].) The same holds if the group G' is compact. If the dual pair is of type II, then the explicit formulas for the Chc [BP06b] show that the conjecture holds. Furthermore, these formulas combined with the equation

$$\frac{1}{x - i0} - \frac{1}{x + i0} = 2\pi i \delta(x) \quad (x \in \mathbb{R}),$$

imply and generalize the results of Adams, [Ada98] and Renard, [Ren98], concerning stable orbital integrals and the theta lift, see section 2.

The plan of this paper is as follows. In Section 1, we provide explicit formulas for the chc and for Chc . In Section 2 we consider sums of the integrals corresponding to various real forms of a dual pair. In Section 3, we prove Theorem 1. In [BP06a], we proved the boundedness of chc . In [BP06b], we proved that the jump relations are satisfied for the dual pairs $(\text{Sp}_{2n}(\mathbb{R}), \text{O}_{1,2})$,

$(O_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$ and $(U_{p,q}, U_{1,1})$. Here we prove a result of reduction and deduce that the jump relations are satisfied for all dual pairs. The proof of Theorem 2 is similar to Theorem 1, and we provide a sketch in Section 4. In Section 5, we define a certain open subset of \tilde{G}' on which *Chc* is an explicitly known smooth function. We prove that on this open subset *Chc* is compatible with the Capelli Harish-Chandra homomorphism (see Theorem 5.3). This implies that the compatibility is satisfied for any regular semisimple element belonging to a fundamental Cartan subgroup. Then, after recalling some classical result of Harish-Chandra (see Subsection 5.2), we prove Theorem 3 using an induction method.

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1. EXPLICIT FORMULAS

Theorem 9 in [BP06b] shows how to compute the Cauchy Harish-Chandra integral on every Cartan subgroup of \tilde{G}' assuming we know how to do it for a compact Cartan subgroup. Thus, let $H' \subseteq G'$ be a compact Cartan subgroup. As a generalized function, the unnormalized Cauchy Harish-Chandra integral satisfies the following identity

$$\widetilde{Chc}(h'g) = \frac{1}{\Theta(\mathfrak{d})} \Theta(\mathfrak{d}h'g) \quad (h' \in \tilde{H}^{reg}, g \in \tilde{G}), \quad (8)$$

where Θ is the character of the oscillator representation ω and $H^{reg} \subseteq H'$ stands for the subset of regular elements. In Theorem 1.2 below we shall give an explicit formula for

$$\int_{\tilde{G}_1} \Theta(h'g) f(g) d\mu(g) \quad (h' \in \tilde{H}^{reg}, f \in \mathcal{D}(\tilde{G}_1)).$$

Since G' has a compact Cartan subgroup, there is a division algebra $\mathbb{D} = \mathbb{R}, \mathbb{C}$ or \mathbb{H} , with an involution ι , and a finite dimensional space V' over \mathbb{D} with a non-degenerate form $(,)'$ which is either ι -hermitian or ι -skew-hermitian, so that G' may be identified with the isometry group of that form. Let

$$V' = V'_0 \oplus V'_1 \oplus \cdots \oplus V'_{n'} \quad (9)$$

be the decomposition into H' -irreducibles over \mathbb{D} . Here $V'_0 = 0$ unless $\mathbb{D} = \mathbb{R}$, the form $(,)'$ is symmetric and $\dim(V')$ is odd. In that case \mathfrak{h}' acts trivially on V'_0 and $\dim(V'_0) = 1$. There is an element $J' \in \mathfrak{h}'$ such that $J'^2 = -1$ on $V'_1 \oplus \cdots \oplus V'_{n'}$. Let J'_j denote the restriction of J' to V'_j , $1 \leq j \leq n'$.

The group G coincides with the isometry group of a form $(,)$ of opposite type to $(,)'$ on a finite dimensional vector space V over \mathbb{D} . The symplectic space $W = \mathrm{Hom}(V, V')$, with the symplectic form given by the formula

$$\langle w', w \rangle = \mathrm{tr}_{\mathbb{D}/\mathbb{R}}(w^* w') \quad (w, w' \in W),$$

where $w^* \in \mathrm{Hom}(V', V)$ is defined by

$$(wv, v')' = (v, w^* v') \quad (v \in V, v' \in V').$$

We have the obvious inclusions: $G, G' \subseteq \mathrm{Sp}(W)$ and $\mathfrak{g}, \mathfrak{g}' \subseteq \mathfrak{sp}(W)$.

If $V'_0 \neq 0$ we choose an element $J \in \mathfrak{g}$ such that $J^2 = -1$ in $\mathrm{End}(V)$ and the restriction of the symmetric form $\langle J, , \rangle$ to the subspace $\mathrm{Hom}(V, V'_0) \subseteq W$ is positive definite: $\langle J, , \rangle_{\mathrm{Hom}(V, V'_0)} > 0$. We shall view W as a complex vector space by

$$iw = \begin{cases} J'(w) & \text{if } w \in \mathrm{Hom}(V, V'_1 + V'_2 + \cdots + V'_{n'}), \\ J(w) & \text{if } w \in \mathrm{Hom}(V, V'_0). \end{cases} \quad (10)$$

Let $\det : \text{End}_{\mathbb{C}}(W) \rightarrow \mathbb{C}$ denote the corresponding determinant and let

$$\widetilde{\text{GL}}_{\mathbb{C}}(W) = \{\tilde{g} = (g, \xi); \xi^2 = \det(g), g \in \text{GL}_{\mathbb{C}}(W)\}.$$

Define

$$\det^{1/2}(\tilde{g}) = \xi, \quad (\tilde{g} = (g, \xi) \in \widetilde{\text{GL}}_{\mathbb{C}}(W)).$$

Let $H'_{\mathbb{C}}, G_{\mathbb{C}} \subseteq \text{GL}_{\mathbb{C}}(W)$ be the complexifications of H' and G respectively. The character Θ extends to a rational function

$$\Theta(\tilde{h}'\tilde{g}) = (-1)^{p-} \frac{\det^{1/2}(\tilde{h}'\tilde{g})}{\det(1 - h'g)} \quad (\tilde{h}' \in \widetilde{H}'_{\mathbb{C}}, \tilde{g} \in \widetilde{G}_{\mathbb{C}}), \quad (11)$$

where $\widetilde{H}'_{\mathbb{C}}, \widetilde{G}_{\mathbb{C}} \subseteq \widetilde{\text{GL}}_{\mathbb{C}}(W)$ are the preimages of the complexifications of H', G , and $2p_-$ is the maximal dimension of real subspace of W on which the symmetric form $\langle J', \cdot, \cdot \rangle$ is negative definite. This follows from Theorem 2.13 and formula (10.10) in [Prz00]. (Indeed, it suffices to consider the pair $G = U_{p,q}, G' = U_1$. Then $n = p + q$. Theorem 2.13 in [Prz00] shows that

$$\lim_{t \rightarrow 0} t^n \Theta(\widetilde{c}_-(tx)) = 2^{-\dim_{\mathbb{R}}(W)} \widetilde{chc}(x) \quad (x \in \mathfrak{g}),$$

where \widetilde{c}_- is the lift of the Cayley transform $c_-(x) = (1+x)(1-x)^{-1}$ such that $\widetilde{c}_-(0)$ is the identity. On the other hand

$$\lim_{t \rightarrow 0} t^n \frac{\det^{1/2}(\widetilde{c}_-(tx))}{\det(1 - c_-(tx))} = \frac{1}{\det(-2)} \frac{1}{\det(x)}.$$

Since, by [Prz00, (10.10)],

$$\widetilde{chc}(x) = (-1)^{p-} 2^n \frac{1}{\det(x)},$$

we see that

$$\Theta(\tilde{g}) \frac{\det(1 - g)}{\det^{1/2}(\tilde{g})} = 2^{-\dim_{\mathbb{R}}(W)} \det(-2) (-1)^{p-} 2^n = (-1)^{n+p} = (-1)^q,$$

and the claim follows.)

Let $H \subseteq G$ be a fundamental Cartan subgroup and let

$$V = V_0 \oplus V_1 \oplus \cdots \oplus V_n \quad (12)$$

be the decomposition into H -irreducibles over \mathbb{D} . Here $V_0 = 0$ unless $\mathbb{D} = \mathbb{R}$, the form (\cdot, \cdot) is symmetric and $\dim(V)$ is odd. In that case \mathfrak{h} acts trivially on V_0 and $\dim(V_0) = 1$. The group H is compact unless $\mathbb{D} = \mathbb{R}$, the form (\cdot, \cdot) is symmetric, $\dim(V)$ is even but the Witt index of (\cdot, \cdot) is odd (i.e. G is isomorphic to $O_{2p+1, 2q+1}$). If H is compact then there is an element $J \in \mathfrak{h}$ (consistent with (10)) whose square equals minus identity on $V_1 + V_2 + \cdots + V_n$. Let $J_j = J|_{V_j}$, $1 \leq j \leq n$. Suppose H is not compact. Then we may assume that $H|_{V_j}$ is compact for each $2 \leq j \leq n$ and that there is $J_1 \in \text{End}(V_1)$ with $J_1^2 = -1$, such that $\mathfrak{h}|_{V_1}$ is conjugate over \mathbb{C} to $\mathbb{R}iJ_1$. As before, let $J = J_1 + J_2 + \cdots + J_n$. In any case, J_1, J_2, \dots, J_n is a basis of the complex vector space $\mathfrak{h}_{\mathbb{C}}$. Let $J_1^*, J_2^*, \dots, J_n^*$ be the dual basis, and let $e_j = iJ_j^*$, $1 \leq j \leq n$. Also, we shall identify

$$V'_j = V_j, \quad J'_j = J_j \quad (1 \leq j \leq n'). \quad (13)$$

In particular $\mathfrak{h}'_{\mathbb{C}} \subseteq \mathfrak{h}_{\mathbb{C}}$. Let

$$W^{\mathfrak{h}'} = \sum_{j=1}^{n'} \text{Hom}(V_j, V_j)^{J_j}.$$

This is a complex subspace of W consisting of all elements which commute with J .

Let $H_{\mathbb{C},1}$ denote the identity component of $H_{\mathbb{C}}$. Then $H_{\mathbb{C},1}$ is isomorphic to

$$\mathfrak{h}_{\mathbb{C}} / \left\{ \sum_{j=1}^n 2\pi x_j J_j \mid x_j \in \mathbb{Z} \right\}.$$

Let $\check{H}_{\mathbb{C},1}$ denote the connected two fold cover of $H_{\mathbb{C},1}$ isomorphic to

$$\mathfrak{h}_{\mathbb{C}}/\left\{\sum_{j=1}^n 2\pi x_j J_j \mid \sum_{j=1}^n x_j J_j \in 2\mathbb{Z}, x_j \in \mathbb{Z}, 1 \leq j \leq n\right\}.$$

Then we have the following covering maps

$$\check{p} : \check{H}_{\mathbb{C},1} \rightarrow \check{H}_{\mathbb{C},1}, \quad \tilde{p} : \tilde{H}_{\mathbb{C},1} \rightarrow H_{\mathbb{C},1}, \quad p = \tilde{p} \circ \check{p} : \check{H}_{\mathbb{C},1} \rightarrow H_{\mathbb{C},1}. \quad (14)$$

Here, \check{p} is either an isomorphism or $\check{H}_{\mathbb{C},1}$ coincides with the direct product $H_{\mathbb{C},1} \times \{\pm 1\}$ and $\check{p} = p$. Similarly, we have

$$\check{p}' : \check{H}'_{\mathbb{C},1} \rightarrow \check{H}'_{\mathbb{C},1}, \quad \tilde{p}' : \tilde{H}'_{\mathbb{C},1} \rightarrow H'_{\mathbb{C},1}, \quad p' = \tilde{p}' \circ \check{p}' : \check{H}'_{\mathbb{C},1} \rightarrow H'_{\mathbb{C},1}. \quad (15)$$

Fix a system Ψ' of positive roots of $\mathfrak{h}'_{\mathbb{C}}$ in $\mathfrak{g}'_{\mathbb{C}}$ and a system Ψ of positive roots of $\mathfrak{h}_{\mathbb{C}}$ in $\mathfrak{g}_{\mathbb{C}}$. Let $\Phi = -\Psi$. Let Z denote the centralizer of $\mathfrak{h}'_{\mathbb{C}}$ in G and let $W(H_{\mathbb{C}}, Z_{\mathbb{C}})$ be the corresponding Weyl group. Let $\Phi(Z) = \Phi \cap \mathfrak{h}'^{\perp}$. This is a system of positive roots of $\mathfrak{h}_{\mathbb{C}}$ in the Lie algebra of $Z_{\mathbb{C}}$. For $h' \in \check{H}'_{\mathbb{C},1}$ and $h \in \check{H}_{\mathbb{C},1}$ define

$$\begin{aligned} \Delta_{\Psi'}(h') &= h'^{\frac{1}{2} \sum_{\alpha \in \Psi'} \alpha} \prod_{\alpha \in \Psi'} (1 - h'^{-\alpha}) \\ \Delta_{\Phi}(h) &= h^{\frac{1}{2} \sum_{\alpha \in \Phi} \alpha} \prod_{\alpha \in \Phi} (1 - h^{-\alpha}) \\ \Delta_{\Phi(Z)}(h) &= h^{\frac{1}{2} \sum_{\alpha \in \Phi(Z)} \alpha} \prod_{\alpha \in \Phi(Z)} (1 - h^{-\alpha}) \\ \det^{1/2}(h)_{\mathbb{W}^{b'}} &= h^{\frac{1}{2} \sum_{j=1}^{n'} e_j}. \end{aligned} \quad (16)$$

The choice of the covering is such that the above definitions make sense. Furthermore,

$$(\det^{1/2}(h)_{\mathbb{W}^{b'}})^2 = \det(p(h))_{\mathbb{W}^{b'}}$$

where $\det(p(h))_{\mathbb{W}^{b'}}$ is the usual determinant of $p(h) \in \text{End}_{\mathbb{C}}(\mathbb{W}^{b'})$.

Define a number $k = -1, 0$ or 1 as follows :

$$k = \begin{cases} -1 & \text{if } (G_{\mathbb{C}}, G'_{\mathbb{C}}) = (\text{GL}_n(\mathbb{C}), \text{GL}_{n'}(\mathbb{C})) \text{ and } n - n' \in 2\mathbb{Z}, \\ & \text{or } (G_{\mathbb{C}}, G'_{\mathbb{C}}) = (\text{O}_{2n+1}(\mathbb{C}), \text{Sp}_{2n'}(\mathbb{C})), \\ 1 & \text{if } (G_{\mathbb{C}}, G'_{\mathbb{C}}) = (\text{Sp}_{2n}(\mathbb{C}), \text{O}_{2n'+1}(\mathbb{C})), \\ 0 & \text{otherwise.} \end{cases}$$

Define a character sign of the Weyl group $W(H_{\mathbb{C}}, G_{\mathbb{C}}) = W(H_{\mathbb{C}})$ by

$$\Delta_{\Phi}(s.h) \frac{\det^{1/2}(s.h)_{\text{Hom}(V, V'_0)}}{\det(1 - p(s.h))_{\text{Hom}(V, V'_0)}} = \text{sign}(s) \Delta_{\Phi}(h) \frac{\det^{1/2}(h)_{\text{Hom}(V, V'_0)}}{\det(1 - p(h))_{\text{Hom}(V, V'_0)}} \quad (s \in W(H_{\mathbb{C}}), h \in \check{H}_{\mathbb{C},1}). \quad (17)$$

The group $W(H_{\mathbb{C}})$ acts on the real span of the J_1, \dots, J_n (which is the same as \mathfrak{h} if H is compact) and is realized as conjugations by elements of $\text{GL}_{\mathbb{C}}(W)$ as in [BP06a].

Proposition 1.1. *There is a constant $v = \pm 1$, which depends only on the choice of the positive root systems Ψ', Φ , such that for $h' \in \check{H}'_{\mathbb{C},1}$ and $h \in \check{H}_{\mathbb{C},1}$,*

$$\begin{aligned} & \det^{k/2}(h')_{\mathbb{W}^{b'}} \Delta_{\Psi'}(h') \Theta(\check{p}(h') \check{p}(h)) \Delta_{\Phi}(h) \\ &= \sum_{s \in W(H_{\mathbb{C}})} (-1)^{p-} \frac{v \text{sign}(s)}{|W(H_{\mathbb{C}}, Z_{\mathbb{C}})|} \frac{\det^{k/2}(s^{-1}.h)_{\mathbb{W}^{b'}} \Delta_{\Phi(Z)}(s^{-1}.h)}{\det(1 - p(h') p(s^{-1}.h))_{\mathbb{W}^{b'}}} \frac{\det^{1/2}(s^{-1}.h)_{W_0}}{\det(1 - p(s^{-1}.h))_{W_0}}, \end{aligned}$$

where $W_0 = \text{Hom}(\sum_{j=n'+1}^n V_j, V'_0)$.

This is verified by the argument used in Appendix B of [BP06a] to prove the corresponding statement for the Lie algebra. The factor $\det^{k/2}$ is necessary for the partial fraction decomposition to work. Unfortunately it was overlooked in [Prz00, (14.5)]. Let

$$U = \begin{cases} V_{\mathbb{C}} & \text{if } \mathbb{D} \neq \mathbb{C}, \\ V & \text{if } \mathbb{D} = \mathbb{C}. \end{cases}$$

Let Ψ_{st}^n denote the family of strongly orthogonal non-compact imaginary roots in Ψ , as in [BP06a]. For each $\mathcal{S} \in \Psi_{st}^n$ there is an element $C(\mathcal{S}) \in \mathrm{GL}_{\mathbb{C}}(U)$ such that the map

$$\mathrm{End}(U) \ni x \rightarrow c(\mathcal{S})(x) = C(\mathcal{S})x C(\mathcal{S})^{-1} \in \mathrm{End}(U),$$

when restricted to $\mathfrak{g}_{\mathbb{C}}$, coincides with the Cayley transform. Let $H(\mathcal{S}) \subseteq G$ be the corresponding Cartan subgroup and let

$$H_{\mathcal{S}} = c(\mathcal{S})^{-1}(H(\mathcal{S})) \subseteq H_{\mathbb{C}}.$$

The map $c(\mathcal{S})$ lifts to the covering, and we shall use the same symbol $c(\mathcal{S})$ to denote these lifts. Let $H_{\mathcal{S},1} = H_{\mathcal{S}} \cap H_{\mathbb{C},1}$ and let $\check{H}_{\mathcal{S},1} \subseteq \check{H}_{\mathbb{C},1}$, $\tilde{H}_{\mathcal{S},1} \subseteq \tilde{H}_{\mathbb{C},1}$ be the corresponding preimages under the covering maps \check{p} , \tilde{p} . Let $\Psi_{\mathcal{S},\mathbb{R}} \subseteq \Psi$ denote the set of the real roots for $H_{\mathcal{S}}$. Define

$$\epsilon_{\Psi_{\mathcal{S},\mathbb{R}}}(h) = \mathrm{sign} \left(\prod_{\alpha \in \Psi_{\mathcal{S},\mathbb{R}}} (1 - h^{-\alpha}) \right) \quad (h \in \check{H}_{\mathcal{S},1}^{\mathrm{reg}}),$$

where $\check{H}_{\mathcal{S},1}^{\mathrm{reg}}$ stands for the set of regular elements in $\check{H}_{\mathcal{S},1}$. Recall the Harish-Chandra orbital integral of a function $f \in \mathcal{D}(\tilde{G}_1)$:

$$\mathcal{H}_{\mathcal{S}} f(h) = \epsilon_{\Psi_{\mathcal{S},\mathbb{R}}}(h) \Delta_{\Psi}(h) \int_{G/H(\mathcal{S})} f(g.c(\mathcal{S})(\tilde{p}(h))) d\mu(gH(\mathcal{S})) \quad (h \in \check{H}_{\mathcal{S},1}^{\mathrm{reg}}).$$

Notice that the function $\Delta_{\Phi}(h)\mathcal{H}_{\mathcal{S}} f(h)$ is constant on the fibers of the covering map \check{p} . Hence the Weyl integration formula for \tilde{G}_1 looks as follows

$$\int_{\tilde{G}_1} f(g) d\mu(g) = \sum_{\mathcal{S} \in \Psi_{st}^n} m_{\mathcal{S}} \int_{\tilde{H}_{\mathcal{S}}} \epsilon_{\Psi_{\mathcal{S},\mathbb{R}}}(h) \Delta_{\Phi}(h) \mathcal{H}_{\mathcal{S}} f(h) d\mu(h),$$

where the $m_{\mathcal{S}}$ are appropriate, uniquely determined, constants.

For a subset $A \subseteq \Psi$ let $\underline{A} = \{j \mid \text{there is } \alpha \in A \text{ such that } \alpha(J_j) \neq 0\}$. For $s \in W(H_{\mathbb{C}})$ and $\mathcal{S} \in \Psi_{st}^n$ define

$$\Gamma_{s,\mathcal{S}} = \{y \in \mathfrak{h} \mid \langle y, \cdot \rangle_{sW^{\mathfrak{b}'} \cap \sum_{j \notin \underline{A}} \mathrm{Hom}(V_j, V')} > 0\},$$

as in [BP06a], and let $E_{s,\mathcal{S}} = \exp(\Gamma_{s,\mathcal{S}}) \subseteq \tilde{H}_{\mathbb{C},1}$. Furthermore, let

$$M_{\mathcal{S}}(s) = (-1)^{p-} \frac{v m_{\mathcal{S}} \mathrm{sign}(s)}{|W(H_{\mathbb{C}}, Z_{\mathbb{C}})|}.$$

Theorem 1.2. *Let $h' \in \check{H}'_1 (= \check{H}'_{\emptyset,1})$ and let $f \in \mathcal{D}(\tilde{G}_1)$. Then*

$$\det^{k/2}(h')_{W^{\mathfrak{b}'}} \Delta_{\Psi'}(h') \int_{\tilde{G}_1} \Theta(\check{p}(h')g) f(g) d\mu(g) = \sum_{s \in W(H_{\mathbb{C}})} \sum_{\mathcal{S} \in \Psi_{st}^n} M_{\mathcal{S}}(s) \lim_{r \in E_{s,\mathcal{S}}, r \rightarrow 1} \int_{\check{H}_{\mathcal{S},1}^{\mathrm{reg}}} \frac{\det^{k/2}(s^{-1}.h)_{W^{\mathfrak{b}'}} \Delta_{\Phi(Z)}(s^{-1}.h)}{\det(1 - p(h')rp(h))_{sW^{\mathfrak{b}'}}} \frac{\det^{1/2}(s^{-1}.h)_{W_0}}{\det(1 - p(s^{-1}.h))_{W_0}} \epsilon_{\Psi_{\mathcal{S},\mathbb{R}}}(h) \mathcal{H}_{\mathcal{S}} f(h) \mu(h).$$

Here the factor $\frac{\Delta_{\Phi(Z)}(s^{-1}.h)}{\det(1 - p(s^{-1}.h))_{W_0}}$ is a smooth function.

This follows from Proposition 1.1 by localization, as in the proof of Theorem 7.3 in [BP06a]. Notice that if G' is an odd real orthogonal group (i.e. $\mathbb{D} = \mathbb{R}$, the form $(\cdot, \cdot)'$ is symmetric and $\dim(V')$ is odd), then $\mathfrak{d} \notin \check{H}'_{\emptyset,1}$. However, the element \mathfrak{d} is in the center of the metaplectic group and therefore we may view it as an element of \tilde{G}_1 .

2. A RELATION WITH STABLE ORBITAL INTEGRALS

Let us define the following two distributions on the unit circle in the complex plane:

$$\frac{1}{1-1_{+1}z} = \lim_{r>1, r \rightarrow 1} \frac{1}{1-rz}, \quad \frac{1}{1-1_{-1}z} = \lim_{r<1, r \rightarrow 1} \frac{1}{1-rz} \quad (z \in \mathbb{C}, |z|=1).$$

Then, as is well known [Hör83, Example 3.1.13],

$$\frac{1}{2\pi} \left(\frac{1}{1-1_{+1}z} - \frac{1}{1-1_{-1}z} \right) = \delta(z), \quad (18)$$

where δ stands for the Dirac delta at the identity.

If $\mathbb{D} = \mathbb{C}$ then the Weyl group $W(\mathbb{H}_{\mathbb{C}}, \mathbb{G}_{\mathbb{C}})$ may be identified with the permutation group Σ_n so that

$$\sigma \cdot J_j = J_{\sigma(j)}, \quad (\sigma \in W(\mathbb{H}_{\mathbb{C}}, \mathbb{G}_{\mathbb{C}}), 1 \leq j \leq n).$$

If $\mathbb{D} \neq \mathbb{C}$, then $W(\mathbb{H}_{\mathbb{C}}, \mathbb{G}_{\mathbb{C}})$ is identified with the semidirect product of Σ_n and $(\mathbb{Z}/2\mathbb{Z})^n$, where $\mathbb{Z}/2\mathbb{Z} = \{0, 1\}$ with the addition modulo 2, so that

$$s \cdot J_j = \hat{\epsilon}_j J_{\sigma(j)}, \quad (s = \sigma \epsilon \in W(\mathbb{H}_{\mathbb{C}}, \mathbb{G}_{\mathbb{C}}), \hat{\epsilon}_j = (-1)^{\epsilon_j}, 1 \leq j \leq n).$$

In any case ($\mathbb{D} = \mathbb{C}$ or $\mathbb{D} \neq \mathbb{C}$) we shall think of elements $s \in W(\mathbb{H}_{\mathbb{C}}, \mathbb{G}_{\mathbb{C}})$ as $s = \sigma \epsilon$ where $\epsilon = 0$ if $\mathbb{D} = \mathbb{C}$.

Let $\text{Hom}(\mathbb{V}_l, \mathbb{V}_j)^0 = \text{Hom}(\mathbb{V}_l, \mathbb{V}_j)^J$ be the subspace of elements that commute with J , and let $\text{Hom}(\mathbb{V}_l, \mathbb{V}_j)^1$ be the subspace of elements that anti-commute with J . (The last space is zero if $\mathbb{D} = \mathbb{C}$.) Let

$$y_s = \sum_{j=1}^n \text{sign}\langle J, \cdot \rangle_{\text{Hom}(\mathbb{V}_{\sigma(j)}, \mathbb{V}_j)^{\epsilon_j}} J_{\sigma(j)} \in \mathfrak{h} \quad (s = \sigma \epsilon \in W(\mathbb{H}_{\mathbb{C}}, \mathbb{G}_{\mathbb{C}})). \quad (19)$$

Here, by definition, the $\text{sign}\langle J, \cdot \rangle_{\text{Hom}(\mathbb{V}_{\sigma(j)}, \mathbb{V}_j)^{\epsilon_j}}$ is equal 1 if this form is positive definite and -1 if it is negative definite.

For $h \in \mathbb{H}$ and $1 \leq j \leq n$ let $h_j = h^{e_j} \in \mathbb{C}$, and similarly for \mathbb{H}' . Then, in terms of Theorem 1.2,

$$\lim_{r \in E_{s, \emptyset}, r \rightarrow 1} \frac{1}{\det(1 - h' r h)_{sW\mathfrak{h}'}} = \prod_{j=1}^{n'} \frac{1}{1 - h'_j (1_{J_{\sigma(j)}^*(y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \quad (h' \in \mathbb{H}'^{reg}, h \in \mathbb{H}).$$

We notice that (18) implies the following identity

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^{n'}} \prod_{j=1}^{n'} \frac{\lambda_j}{1 - h'_j (1_{\lambda_j J_{\sigma(j)}^*(y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \\ &= \prod_{j=1}^{n'} (-\hat{\epsilon}_j) \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(y_s) \cdot \prod_{j=1}^{n'} \left(\frac{1}{1 - h'_j 1_{+1} h_{\sigma(j)}^{-\hat{\epsilon}_j}} - \frac{1}{1 - h'_j 1_{-1} h_{\sigma(j)}^{-\hat{\epsilon}_j}} \right) \\ &= (2\pi)^{n'} \cdot \prod_{j=1}^{n'} (-\hat{\epsilon}_j) \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(y_s) \cdot \delta(h'(s^{-1} \cdot h)^{-1}), \end{aligned} \quad (20)$$

where δ stands for the Dirac delta at the identity of \mathbb{H}' . Furthermore,

$$\lim_{r \in E_{s, \mathcal{S}}, r \rightarrow 1} \frac{1}{\det(1 - h' r h)_{sW\mathfrak{h}'}} = \prod_{j=1, j \notin \mathcal{S}}^{n'} \frac{1}{1 - h'_j (1_{J_{\sigma(j)}^*(y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \prod_{j=1, j \in \mathcal{S}}^{n'} \frac{1}{1 - h'_j h_{\sigma(j)}^{-\hat{\epsilon}_j}}, \quad (21)$$

and if $\mathcal{S} \neq \emptyset$, then

$$\sum_{\lambda \in \{\pm 1\}^{n'}} \prod_{j=1, j \notin \mathcal{S}}^{n'} \frac{\lambda_j}{1 - h'_j (1_{\lambda_j J_{\sigma(j)}^*(y_s)} h_{\sigma(j)})^{-\hat{\epsilon}_j}} \prod_{j=1, j \in \mathcal{S}}^{n'} \frac{\lambda_j}{1 - h'_j h_{\sigma(j)}^{-\hat{\epsilon}_j}} = 0. \quad (22)$$

The generalized function $\Theta(h'g)$, (8), and the sets $E_{s,S}$ of Theorem 1.2 depend on the form $(,)'$ (and the form $(,)$). Let $\Theta^\lambda(h'g)$ and $E_{s,S}^\lambda$ denote the corresponding objects for the form

$$\lambda_1(,)'|_{V_1} + \lambda_2(,)'|_{V_2} + \cdots + \lambda_{n'}(,)'|_{V_{n'}}, \quad (23)$$

where $\lambda = (\lambda_1, \dots, \lambda_{n'}) \in \{\pm 1\}^{n'}$.

Assume from now on that $n' = n$. Then $H' = H$ via the identification (13), and $\check{H}'_1 = \check{H}_1$. The formulas (20) and (22) imply that, in terms of Theorem 1.2,

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^{n'}} \lambda_1 \lambda_2 \cdots \lambda_{n'} \det^{k/2}(h')_{W^{b'}} \Delta_{\Psi'}(h') \int_{\check{G}_1} \Theta^\lambda(\check{p}(h')g) f(g) d\mu(g) \\ = & \sum_{s \in W(\mathbb{H}_\mathbb{C}, \mathbb{G}_\mathbb{C})} M_\emptyset(s) \sum_{\lambda \in \{\pm 1\}^{n'}} \lambda_1 \lambda_2 \cdots \lambda_{n'} \\ & \lim_{r \in E_{s,\emptyset}^\lambda, r \rightarrow 1} \int_{\check{H}_1} \frac{1}{\det(1 - p(h')rp(h))_{sW^{b'}}} \det^{k/2}(s^{-1}.h)_{W^b} \mathcal{H}_\emptyset f(h) d\mu(h) \\ = & \sum_{s \in W(\mathbb{H}_\mathbb{C}, \mathbb{G}_\mathbb{C})} M_\emptyset(s) (2\pi)^{n'} \cdot \prod_{j=1}^{n'} (-\hat{\epsilon}_j) \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(y_s) \\ & \int_{\check{H}_1} \delta(p(h'(s^{-1}.h)^{-1})) \det^{k/2}(s^{-1}.h)_{W^{b'}} \mathcal{H}_\emptyset f(h) d\mu(h) \\ = & \det^{k/2}(h')_{W^b} \mu(H') (-1)^{p_- + n} \frac{v}{|W(\mathbb{H}, \mathbb{G})|} \sum_{s \in W(\mathbb{H}_\mathbb{C}, \mathbb{G}_\mathbb{C})} \text{sign}(s) \prod_{j=1}^n \hat{\epsilon}_j \cdot \prod_{j=1}^{n'} J_{\sigma(j)}^*(y_s) \cdot \mathcal{H}_\emptyset f(s.h'). \end{aligned} \quad (24)$$

Suppose, from now on, that the form $(,)'$ is hermitian. Then the element $y_{\sigma\epsilon}$ (19) does not depend on the ϵ . Moreover,

$$\prod_{j=1}^{n'} J_{\sigma(j)}^*(y_s) = (-1)^{-n+p_-(W^{b'})},$$

where $2p_-(W^{b'})$ is the maximal dimension of a subspace of $W^{b'}$ on which the symmetric form $\langle J', \rangle$ is negative definite.

Under our assumptions we have the identification $W(H'_\mathbb{C}, G'_\mathbb{C}) = W(H_\mathbb{C}, G_\mathbb{C})$. Let us denote both groups by $W(H_\mathbb{C})$. Furthermore, for $s = \sigma\epsilon \in W(H_\mathbb{C})$,

$$\text{sign}(s) \prod_{j=1}^n \hat{\epsilon}_j = \text{sign}'(s),$$

where the sign' is defined by $\Delta_{\Psi'}(s.h') = \text{sign}'(s) \Delta_{\Psi'}(h')$.

For $\lambda \in \{\pm 1\}^{n'}$ define

$$q(\lambda) = (-1)^{p_- + p_-(W^{b'})} \frac{v}{|W(H_\mathbb{C})|} \lambda_1 \lambda_2 \cdots \lambda_{n'}.$$

Then (24) shows that

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^{n'}} q(\lambda) \frac{1}{\mu(H')} \int_{\check{G}} \Theta^\lambda(\check{p}(h')g) f(g) d\mu(g) \\ & = \frac{1}{|W(\mathbb{H}, \mathbb{G})| |W(H_\mathbb{C})|} \sum_{s \in W(H_\mathbb{C})} \frac{1}{\Delta_{\Psi'}(s.h')} \mathcal{H}_\emptyset f(s.h'). \end{aligned} \quad (26)$$

This function is clearly $W(H_\mathbb{C})$ -invariant.

Let $\gamma \in \mathfrak{h}^* = \mathfrak{h}'^*$ be a regular element. For $h \in \check{H}^{reg}$ define

$$F(h) = \frac{\sum_{s \in W(H_\mathbb{C})} \text{sign}'(s) h^{s\gamma}}{\Delta_{-\Psi'}(h)}.$$

Then (26) shows that

$$\begin{aligned} & \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 F(h) \sum_{\lambda \in \{\pm 1\}^n} q(\lambda) \frac{1}{\mu(\mathbf{H}')} \int_{\tilde{\mathbf{G}}} \Theta^\lambda(\check{p}(h)g) f(g) d\mu(g) d\mu(h) \\ &= \frac{1}{|W(\mathbf{H}, \mathbf{G})|} \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 \frac{1}{|W(\mathbf{H}_\mathbb{C})|} \sum_{s \in W(\mathbf{H}_\mathbb{C})} \left(F(s.h) \frac{\Delta_{-\Psi'}(s.h)}{\Delta_{-\Psi}(s.h)} \right) \int_{\mathbf{G}/\mathbf{H}} f(g.\check{p}(s.h)) d\mu(g\mathbf{H}) d\mu(h). \end{aligned} \quad (27)$$

The group Σ_n acts on the set $\{\pm 1\}^n$ so that

$$\Theta^{\sigma\lambda}(hg) = \Theta^\lambda((\sigma^{-1}.h)g) \quad (\sigma \in \Sigma_n, \lambda \in \{\pm 1\}^n, h \in \tilde{\mathbf{H}}^{reg}, g \in \tilde{\mathbf{G}}).$$

Hence, the left hand side of (27) may be rewritten as

$$\begin{aligned} & \sum_{\lambda \in \{\pm 1\}^n} q(\lambda) \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(\mathbf{H}')} \int_{\tilde{\mathbf{G}}} \Theta^\lambda(\check{p}(h)g) f(g) d\mu(g) d\mu(h) \\ &= \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} \sum_{[\sigma] \in \Sigma_n / Stab_{\Sigma_n}(\lambda)} q(\sigma\lambda) \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(\mathbf{H}')} \int_{\tilde{\mathbf{G}}} \Theta^{\sigma\lambda}(\check{p}(h)g) f(g) d\mu(g) d\mu(h) \\ &= \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} \sum_{[\sigma] \in \Sigma_n / Stab_{\Sigma_n}(\lambda)} q(\lambda) \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(\mathbf{H}')} \int_{\tilde{\mathbf{G}}} \Theta^\lambda(\check{p}(h)g) f(g) d\mu(g) d\mu(h) \end{aligned} \quad (28)$$

Let G'_λ be the isometry group of the form (23). Then

$$|\Sigma_n / Stab_{\Sigma_n}(\lambda)| = \frac{|W(\mathbf{H}_\mathbb{C})|}{|W(\mathbf{H}', G'_\lambda)|}.$$

Hence, (28) is equal to

$$\sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} q(\lambda) \frac{|W(\mathbf{H}_\mathbb{C})|}{|W(\mathbf{H}', G'_\lambda)|} \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 F(h) \frac{1}{\mu(\mathbf{H}')} \int_{\tilde{\mathbf{G}}} \Theta^\lambda(\check{p}(h)g) f(g) d\mu(g) d\mu(h) \quad (29)$$

We know from Theorem 2, there are functions $f_\lambda \in \mathcal{D}(\tilde{\mathbf{G}}'_\lambda)$ such that

$$\frac{1}{\mu(\mathbf{H}')} \int_{\tilde{\mathbf{G}}} \Theta^\lambda(\check{p}(h)g) f(g) d\mu(g) = \int_{\tilde{\mathbf{G}}'/\tilde{\mathbf{H}}'} f_\lambda(g.\check{p}(h)) d\mu(g\mathbf{H}'). \quad (30)$$

Furthermore,

$$\begin{aligned} F(h) &= \sum_{[s] \in W(\mathbf{H}', G'_\lambda) \setminus W(\mathbf{H}_\mathbb{C})} F_{\lambda, [s]}(h), \quad \text{where} \\ F_{\lambda, [s]}(h) &= \frac{\sum_{s' \in W(\mathbf{H}', G'_\lambda)} \text{sign}'(s's) h^{s's\gamma}}{\Delta_{-\Psi'}(h)}. \end{aligned} \quad (31)$$

Hence, (29) is equal to

$$\begin{aligned} & \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} q(\lambda) |W(\mathbf{H}_\mathbb{C})| \sum_{[s] \in W(\mathbf{H}', G'_\lambda) \setminus W(\mathbf{H}_\mathbb{C})} \\ & \frac{1}{|W(\mathbf{H}', G'_\lambda)|} \int_{\tilde{\mathbf{H}}^{reg}} |\Delta_{\Psi'}(h)|^2 F_{\lambda, [s]}(h) \int_{\tilde{\mathbf{G}}'/\tilde{\mathbf{H}}'} f_\lambda(g.\check{p}(h)) d\mu(g\mathbf{H}'). \end{aligned} \quad (32)$$

We may assume that the forms $(\ , \)'$ and $\langle J \ , \ \rangle$ are positive definite. Then $(-1)^{p-+p-(W^{\mathfrak{h}'})} = 1$. Thus we see from (32) that the equality (27) may be rewritten as

$$\begin{aligned} & \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} v \lambda_1 \lambda_2 \cdots \lambda_n \sum_{[s] \in W(\mathfrak{H}', \mathfrak{G}'_\lambda) \setminus W(\mathfrak{H}_\mathbb{C})} \\ & \frac{1}{|W(\mathfrak{H}', \mathfrak{G}'_\lambda)|} \int_{\tilde{\mathfrak{H}}'^{reg}} |\Delta_{\Psi'}(h)|^2 F_{\lambda, [s]}(h) \int_{\tilde{\mathfrak{G}}' / \tilde{\mathfrak{H}}'} f_\lambda(g \cdot \check{p}(h)) d\mu(g\mathfrak{H}') \\ & = \frac{1}{|W(\mathfrak{H}, \mathfrak{G})|} \int_{\mathfrak{H}'^{reg}} |\Delta_\Psi(h)|^2 \frac{1}{|W(\mathfrak{H}_\mathbb{C})|} \\ & \sum_{s \in W(\mathfrak{H}_\mathbb{C})} \left(F(s, h) \frac{\Delta_{-\Psi'}(s, h)}{\Delta_{-\Psi}(s, h)} \right) \int_{\mathfrak{G}/\mathfrak{H}} f(g \cdot \check{p}(s, h)) d\mu(g\mathfrak{H}) d\mu(h). \quad (33) \end{aligned}$$

If $\mathbb{D} = \mathbb{R}$ and $\dim(\mathfrak{V}') = 2n + 1$ then

$$\lambda_1 \lambda_2 \cdots \lambda_n = (-1)^{\frac{1}{2} \dim(\mathfrak{G}'_\lambda / \mathfrak{K}'_\lambda)},$$

where $\mathfrak{K}'_\lambda \subseteq \mathfrak{G}'_\lambda$ is a maximal compact subgroup. In this case (33) simplifies further to

$$\begin{aligned} & \sum_{[\lambda] \in \{\pm 1\}^n / \Sigma_n} v (-1)^{\frac{1}{2} \dim(\mathfrak{G}'_\lambda / \mathfrak{K}'_\lambda)} \sum_{[s] \in W(\mathfrak{H}', \mathfrak{G}'_\lambda) \setminus W(\mathfrak{H}_\mathbb{C})} \\ & \frac{1}{|W(\mathfrak{H}', \mathfrak{G}'_\lambda)|} \int_{\tilde{\mathfrak{H}}'^{reg}} |\Delta_{\Psi'}(h)|^2 F_{\lambda, [s]}(h) \int_{\tilde{\mathfrak{G}}' / \tilde{\mathfrak{H}}'} f_\lambda(g \cdot \check{p}(h)) d\mu(g\mathfrak{H}') \\ & = \frac{1}{|W(\mathfrak{H}, \mathfrak{G})|} \int_{\mathfrak{H}'^{reg}} |\Delta_\Psi(h)|^2 \left(F(h) \frac{\Delta_{-\Psi'}(h)}{\Delta_{-\Psi}(h)} \right) \sum_{s \in W(\mathfrak{H}_\mathbb{C})} \int_{\mathfrak{G}/\mathfrak{H}} f(g \cdot \check{p}(s, h)) d\mu(g\mathfrak{H}) d\mu(h). \quad (34) \end{aligned}$$

Thus, in this case, the weighted sum of the Cauchy Harish-Chandra integrals (26) coincides with the transfer map studied by Adams [Ada98, Def. 4.5] and Renard [Ren98].

3. PROOF OF THEOREM 1

3.1. Properties of Cauchy Harish-Chandra integral on the Lie algebra. Let $\mathfrak{h}' \subseteq \mathfrak{g}'$ be a θ -stable Cartan subalgebra. Denote by $\Delta_I(\mathfrak{g}'_\mathbb{C}, \mathfrak{h}'_\mathbb{C})$ the set of imaginary roots of \mathfrak{h}' in $\mathfrak{g}'_\mathbb{C}$. Fix a positive system $\Psi \subseteq \Delta_I(\mathfrak{g}'_\mathbb{C}, \mathfrak{h}'_\mathbb{C})$. For $\psi \in \mathcal{S}(\mathfrak{g})$ and for $x' \in \mathfrak{h}'^{reg}$ define

$$chc_{W, \Psi, \mathfrak{h}'}(\psi)(x') = \prod_{\alpha \in \Psi} \frac{\alpha(x')}{|\alpha(x')|} |\det(\text{ad}(x'))_{\mathfrak{g}'/\mathfrak{h}'}|^{1/2} \int_{\mathfrak{g}} chc_W(x' + x) \psi(x) d\mu(x). \quad (35)$$

Let $\mathfrak{h}'_{In-reg} = \{x' \in \mathfrak{h}' \mid \alpha(x') \neq 0 \text{ for all } \alpha \in \Psi\}$.

We proved in [BP06a] that the function $chc_{W, \Psi, \mathfrak{h}'}$ is smooth on \mathfrak{h}'^{reg} and for any $w \in \text{Sym}(\mathfrak{h}'_\mathbb{C})$ the derivative $\partial(w)chc_{W, \Psi, \mathfrak{h}'}$ is locally bounded. The remaining property to be proved is the jump relation (see (36)).

Fix a single non-compact imaginary root $\alpha \in \Psi$. Let $x' \in \mathfrak{h}'$ be such that $\alpha(x') = 0$ and the derived Lie algebra $[\mathfrak{g}'^{x'}, \mathfrak{g}'^{x'}]$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$. This means that x' is a sub-regular element attached to the non-compact imaginary root α . Let $H_\alpha \in i\mathfrak{h}'$, $X_{\pm\alpha} \in \mathfrak{g}'_{\mathbb{C}, \pm\alpha}$ be such that

$$[X_\alpha, X_{-\alpha}] = H_\alpha, [H_\alpha, X_{\pm\alpha}] = \pm 2X_{\pm\alpha}, \overline{X}_\alpha = X_{-\alpha},$$

where $X \rightarrow \overline{X}$ stands for the conjugation with respect to the real form $\mathfrak{g}' \subseteq \mathfrak{g}'_\mathbb{C}$. In particular, $(X_\alpha, H_\alpha, X_{-\alpha})$ is a \mathfrak{sl}_2 -triple. Then, we have the decomposition

$$\mathfrak{h}' = \mathbb{R}iH_\alpha \oplus \ker(\alpha).$$

Let

$$c(\alpha) = \exp(-i\frac{\pi}{4} \text{ad}(X_\alpha + X_{-\alpha})) \in \text{End}(\mathfrak{g}'_\mathbb{C}).$$

Then

$$c(\alpha)(\mathfrak{h}'_\mathbb{C}) \cap \mathfrak{g}' = \mathbb{R}s_\alpha H_\alpha \oplus \ker \alpha$$

is another Cartan subalgebra of \mathfrak{g}' denoted by \mathfrak{h}'_α . Let $\Psi \cap \alpha^\perp = \{\beta \in \Psi \mid \beta(H_\alpha) = 0\}$. Then

$$\Psi \cap \alpha^\perp \circ c(\alpha)^{-1}$$

is a system of positive imaginary roots of \mathfrak{h}'_α in $\mathfrak{g}'_\mathbb{C}$ denoted Ψ_α . Set

$$d(\alpha) = \begin{cases} 2 & \text{if the reflection with respect to } \alpha \text{ is realized by an element of } \widetilde{G}'^{x'} \\ 1 & \text{otherwise.} \end{cases}$$

Let $w \in \text{Sym}(\mathfrak{h}'_\mathbb{C})$. Put

$$\langle \partial(w)chc_{\mathcal{W}, \Psi, \mathfrak{h}'(\psi)} \rangle(x') = \lim_{t \rightarrow 0_+} \partial(w)chc_{\mathcal{W}, \Psi, \mathfrak{h}'(\psi)}(x' + tiH_\alpha) - \lim_{t \rightarrow 0_+} \partial(w)chc_{\mathcal{W}, \Psi, \mathfrak{h}'(\psi)}(x' - tiH_\alpha)$$

We need to prove the following jump relation

$$\langle \partial(w)chc_{\mathcal{W}, \Psi, \mathfrak{h}'(\psi)} \rangle(x') = \text{id}(x')\partial(s_\alpha w)chc_{\mathcal{W}, \Psi_\alpha, \mathfrak{h}'_\alpha(\psi)}(x'). \quad (36)$$

3.2. Some useful facts.

Lemma 3.1. *Suppose that H is a compact Cartan subgroup of G . Then $x \in \mathfrak{h}$ is not annihilated by any non-compact imaginary roots if and only if G^x is compact.*

Proof. Recall the Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}.$$

Then

$$\mathfrak{g}_\mathbb{C} = \mathfrak{k}_\mathbb{C} \oplus \mathfrak{p}_\mathbb{C} = \bigoplus_\alpha \mathfrak{k}_{\mathbb{C}, \alpha} \oplus \bigoplus_\beta \mathfrak{p}_{\mathbb{C}, \beta},$$

where the α 's are the compact roots and the β 's are the non-compact roots. Thus

$$\mathfrak{g}_\mathbb{C}^x = \mathfrak{h}_\mathbb{C} \oplus \bigoplus_{\alpha(x)=0} \mathfrak{k}_{\mathbb{C}, \alpha} \oplus \bigoplus_{\beta(x)=0} \mathfrak{p}_{\mathbb{C}, \beta},$$

and the lemma follows. \square

Lemma 3.2. *Let $\mathfrak{h} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let*

$$\mathbb{V} = \bigoplus_j \mathbb{V}_j$$

be the decomposition into \mathfrak{h} -isotypic components over \mathbb{D} . Let $x \in \mathfrak{h}$. Then x is annihilated by precisely one non-compact imaginary root of \mathfrak{h} in $\mathfrak{g}_\mathbb{C}$ if and only if G^x is not compact and either

- (1) *there is exactly one pair $j < k$ such that $\text{eig}(x|_{\mathbb{V}_j}) = \text{eig}(x|_{\mathbb{V}_k})$, and $x|_{\mathbb{V}_l} \neq 0$ for all l if $\mathbb{D} = \mathbb{R}$ and the form $(\ , \)$ is skew-symmetric, or*
- (2) *$\mathbb{D} = \mathbb{R}$, the form $(\ , \)$ is skew-symmetric, the sets $\text{eig}(x|_{\mathbb{V}_j})$ are distinct and there is exactly one l such that $x|_{\mathbb{V}_l} = 0$.*

Here $\text{eig}(x)$ stands for the set of the eigenvalues of x .

Proof. There is $t \in G$ such that $\theta = \text{Ad}(t)$. Since $x = \theta(x)$, t preserves the decomposition of \mathbb{V} . More precisely, if the form $(\ , \)$ is hermitian, then $t|_{\mathbb{V}_j} = \epsilon_j I_{\mathbb{V}_j}$, where $\epsilon_j = 1$ if $(\ , \)_{\mathbb{V}_j} > 0$, and $\epsilon_j = -1$ if $(\ , \)_{\mathbb{V}_j} < 0$. If the form $(\ , \)$ is skew-hermitian, then t is a positive compatible complex structure on \mathbb{V} and there are real numbers r_j such that $x|_{\mathbb{V}_j} = r_j t|_{\mathbb{V}_j}$ for all j . As an \mathfrak{h} -module,

$$\mathfrak{p} = {}^t \mathfrak{g} = \bigoplus_j {}^t \mathfrak{g}(\mathbb{V}_j) \oplus \bigoplus_{j < k} {}^t \text{Hom}(\mathbb{V}_j, \mathbb{V}_k).$$

Here ${}^tE \subseteq F$ stands for the anticommutant of E in F , i.e. ${}^tE = \{y \in F; xy + yx = 0 \text{ for all } x \in E\}$. Suppose the form $(\ , \)$ is hermitian. Then ${}^t\mathfrak{g}(\mathbf{V}_j) = 0$ for all j . Moreover, ${}^t\text{Hom}(\mathbf{V}_j, \mathbf{V}_k) \neq 0$ if and only if $\epsilon_j + \epsilon_k = 0$, and in this case ${}^t\text{Hom}(\mathbf{V}_j, \mathbf{V}_k) = \text{Hom}(\mathbf{V}_j, \mathbf{V}_k)$. Thus

$$\mathfrak{p} = \bigoplus_{j < k, \epsilon_j + \epsilon_k = 0} \text{Hom}(\mathbf{V}_j, \mathbf{V}_k).$$

Let α be a non-compact imaginary root. Then there is exactly one pair $j < k$, with $\epsilon_j + \epsilon_k = 0$, such that α is an eigen-character of \mathfrak{h} in $\text{Hom}(\mathbf{V}_j, \mathbf{V}_k)_{\mathbb{C}}$, i.e. $\text{Hom}(\mathbf{V}_j, \mathbf{V}_k)_{\mathbb{C}, \alpha} \neq 0$. Then $\alpha(x) = 0$ is equivalent to

$$\det(\text{ad}(x))_{\text{Hom}(\mathbf{V}_j, \mathbf{V}_k)} = 0,$$

which, by a case by case verification, is equivalent to

$$\text{eig}(x|_{\mathbf{V}_j}) = \text{eig}(x|_{\mathbf{V}_k}).$$

Suppose the form $(\ , \)$ is skew-hermitian. We may assume that $\mathbb{D} \neq \mathbb{C}$. If α is a non-compact imaginary root, then either

$${}^t\text{Hom}(\mathbf{V}_j, \mathbf{V}_k)_{\mathbb{C}, \alpha} \neq 0, \text{ for some } j < k,$$

or

$${}^t\mathfrak{g}(\mathbf{V}_l)_{\mathbb{C}, \alpha} \neq 0, \text{ for some } l.$$

Notice that

$$\text{ad}(x)_{\mathfrak{h}}|_{\text{Hom}(\mathbf{V}_j, \mathbf{V}_k)} = (r_j + r_k) \text{ad}\left(\frac{1}{2}t\right)_{\mathfrak{h}}|_{\text{Hom}(\mathbf{V}_j, \mathbf{V}_k)},$$

and

$$\text{ad}(x)_{\mathfrak{h}}|_{\mathfrak{g}(\mathbf{V}_l)} = r_l \text{ad}(t)_{\mathfrak{h}}|_{\mathfrak{g}(\mathbf{V}_l)}.$$

This implies our lemma. \square

Corollary 3.3. *Let $\mathfrak{h} \subseteq \mathfrak{k}$ be a Cartan subalgebra of $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Let $x \in \mathfrak{h}$ be such that x is annihilated by precisely one non-compact imaginary root of \mathfrak{h} in $\mathfrak{g}_{\mathbb{C}}$. Then the space \mathbf{V} has a direct sum decomposition*

$$\mathbf{V} = \mathbf{V}_0 \oplus \mathbf{V}_1 \oplus \mathbf{V}_2 \oplus \cdots \oplus \mathbf{V}_m,$$

such that $[\mathfrak{g}^x|_{\mathbf{V}_0}, \mathfrak{g}^x|_{\mathbf{V}_0}]$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and for each $j \geq 1$ the element (x, \mathbf{V}_j) is indecomposable and $\mathfrak{g}^x|_{\mathbf{V}_j} = \mathfrak{h}|_{\mathbf{V}_j}$.

Moreover the sets

$$\text{eig}(x|_{\mathbf{V}_j}) \quad (j = 0, 1, 2, \dots, m)$$

are disjoint.

If (x, \mathbf{V}_0) is indecomposable, then $\mathbb{D} = \mathbb{R}$, the form $(\ , \)$ is skew-symmetric and $x|_{\mathbf{V}_0} = 0$. In this case $\mathfrak{g}^x|_{\mathbf{V}_0} = \mathfrak{g}(\mathbf{V}_0)$ is isomorphic to $\mathfrak{sp}_2(\mathbb{R})$.

Suppose the element (x, \mathbf{V}_0) is decomposable. Then (x, \mathbf{V}_0) is the sum of two distinct indecomposables. If $x|_{\mathbf{V}_0} \neq 0$ then both components are non-zero and $\mathfrak{g}^x|_{\mathbf{V}_0} = \mathfrak{u}_{1,1}$. If $x|_{\mathbf{V}_0} = 0$ then $\mathbb{D} = \mathbb{R}$, the form $(\ , \)$ is symmetric, $(\ , \)_{\mathbf{V}_0}$ has signature $(2, 1)$ or $(1, 2)$ and $\mathfrak{g}^x|_{\mathbf{V}_0} = \mathfrak{g}(\mathbf{V}_0)$ is isomorphic to $\mathfrak{so}(1, 2)$.

3.3. The reduction.

Proposition 3.4. *Suppose H' is a compact Cartan subgroup of G' . Let $\psi \in \mathcal{D}(\mathfrak{g})$. Then the function*

$$\mathfrak{h}'_{In-reg} \ni x' \rightarrow \text{chc}(\psi)(x') = \int_{\mathfrak{g}} \text{chc}(x' + x) \psi(x) d\mu(x) \ni \mathbb{C}$$

is smooth.

Proof. This may be done via a wave front set computation as in Lemma 15.6, [Prz00], or explicitly as follows.

Fix $x' \in \mathfrak{h}'_{In-reg}$. It will suffice to consider the function $chc(\psi)(x')$ for x' in some small neighborhood of x' . Furthermore we may assume that ψ is supported in an arbitrarily small completely invariant neighborhood of a semisimple point $x \in \mathfrak{g}$, which belongs to the singular support of $chc(x'+)$. Let

$$V' = \bigoplus_j V'_j \quad \text{and} \quad V = \bigoplus_k V_k$$

be the isotypic decompositions with respect to x' and x respectively. It is not difficult to check that the sets $eig(x'|_{V'_j})$ are disjoint. For each j , let \tilde{V}_j be the sum of all the V_k such that $eig(x'|_{V'_j}) = eig(x|_{V_k})$. We arrange the indices so that $\tilde{V}_j \neq 0$ if and only if $j = 1, 2, 3, \dots, m$. Then

$$W = \bigoplus_{j=1}^m \ker(x' + x) \cap \text{Hom}(V'_j, \tilde{V}_j) \oplus \ker(x' + x)^\perp.$$

Let $\mathcal{U}' \subseteq \mathfrak{g}^{x'}$ and $\mathcal{U} \subseteq \mathfrak{g}^x$ be slices through x' and x respectively. Then for $x' \in \mathcal{U}'$ and for $x \in \mathcal{U}$,

$$chc_W(x' + y) = \prod_{j=1}^m chc_{\ker(x'+x) \cap \text{Hom}(V'_j, \tilde{V}_j)}(x' + y) chc_{\ker(x'+x)^\perp}(x' + y),$$

where the last factor is a smooth function. Since, by Lemma 3.1, $G^{x'}$ is compact, the above decomposition of chc_W reduces the proof to the case when G' is compact. In this case,

$$\int_{\mathfrak{g}} \widetilde{chc}(x' + x) \psi(x) d\mu(x) = \int_{\mathfrak{g}} \chi\left(\frac{1}{4}\langle xw, w \rangle\right) \phi(w) dw,$$

where $\phi \in \mathcal{S}(W)$ is the pullback via the moment map $W \rightarrow \mathfrak{g}^*$ of a Fourier transform of ψ . (Here the fact that ϕ is rapidly decreasing follows from the compactness of G' .) In particular, it is clear that the function in question is smooth. \square

Proof. [of Theorem 1] Theorem 1 holds for pairs of type II, as was observed by Bernon in [Ber01]. This is a consequence of an explicit formula [Prz00, Prop. 7.21]. Thus we may assume that the pair (G, G') is of type I. Then G, G' are the isometry groups $G = G(V, (\cdot, \cdot))$ and let $G' = G(V', (\cdot, \cdot))$, as in section 1. Recall the θ -stable Cartan subgroup $H' = T'A' \subseteq G'$. Let $V'_c \subseteq V'$ be the subspace on which A' acts trivially, and let $V'_s = V'_c^\perp$. Then V'_s has a complete polarization

$$V'_s = X' \oplus Y'$$

preserved by H' . We may and do assume that $V'_s \subseteq V$ and that the above is also a complete polarization with respect to the form (\cdot, \cdot) . Let $U = V'^\perp \subseteq V$. Then

$$V = V'_s \oplus U.$$

The above decompositions induce embeddings:

$$\begin{aligned} \text{GL}(X') \times G(U) &\subseteq G, \\ \mathfrak{n}' = \text{Hom}(X', V'_c) + \text{Hom}(X', Y') \cap \mathfrak{g}' &\subseteq \mathfrak{g}', \\ \mathfrak{n} = \text{Hom}(X', U) + \text{Hom}(X', Y') \cap \mathfrak{g} &\subseteq \mathfrak{g}. \end{aligned} \tag{37}$$

We assume that the subgroup $\text{GL}(X') \times G(U) \subseteq G$ is preserved by θ . Let $W_c = \text{Hom}(V'_c, U)$. Then, by Corollary 8 of [BP06b], there is a non-zero constant

$$c = \frac{\gamma(V, V', X')}{\eta(V, V', X')},$$

such that, for $\psi \in \mathcal{S}(\mathfrak{g})$, we have

$$\begin{aligned} \text{chc}_{\mathcal{W}, \Psi, \mathfrak{h}'}(\psi)(x') &= c \prod_{\alpha \in \Psi(\mathfrak{gl}(X'))} \frac{\alpha(x')}{|\alpha(x')|} |\det(\text{ad}(x'))_{\mathfrak{gl}(X')/\mathfrak{h}'|_{X'}}|^{1/2} \\ &\int_{\text{GL}(X')/H'|_{X'}} \int_{\mathfrak{g}(\mathbb{U})} \prod_{\alpha \in \Psi(\mathfrak{g}(V'_c))} \frac{\alpha(x')}{|\alpha(x')|} \text{chc}_{\mathcal{W}_c}(x'|_{V'_c} + y) \psi_n^K(g \cdot x'|_{X'} + y) d\mu(y) d\mu(gH'|_{X'}), \end{aligned} \quad (38)$$

where, for the purpose of normalization of the measure μ , $H'|_{X'} \subseteq G' \subseteq \text{Sp}(\mathcal{W})$. Moreover, Ψ is the disjoint union of $\Psi(\mathfrak{gl}(X')) \subseteq \Delta_I(\mathfrak{gl}(X')_{\mathbb{C}}, \mathfrak{h}'_{X', \mathbb{C}})$ and $\Psi(\mathfrak{g}(V'_c)) \subseteq \Delta_I(\mathfrak{g}(V'_c)_{\mathbb{C}}, \mathfrak{h}'_{V'_c, \mathbb{C}})$, and the integral over $\mathfrak{g}(\mathbb{U})$ is not there if $\mathcal{W}_c = 0$. If $\mathcal{W}_c \neq 0$ and \mathfrak{h}' acts trivially on \mathcal{W}_c (or equivalently on V'_c) then the $\text{chc}_{\mathcal{W}_c}(x'|_{V'_c} + y)$ is replaced by $\text{chc}_{\mathcal{W}_c}(y)$.

If $\text{chc}_{\mathcal{W}_c}(y) = 0$ or $\text{chc}_{\mathcal{W}_c}(y) \neq 0$ but $\mathfrak{h}'|_{V'_c} = 0$ then we see from (38) that $\text{chc}_{\mathcal{W}, \Psi, \mathfrak{h}'}(\psi)$ is an orbital integral, so there is nothing to prove.

If $\mathfrak{h}'|_{V'_c} \neq 0$, then $\text{chc}_{\mathcal{W}, \Psi, \mathfrak{h}'}(\psi)$ is essentially the tensor product of an orbital integral for $\mathfrak{gl}(X')$ and $\text{chc}_{\mathcal{W}, \Psi(\mathfrak{g}(V'_c)), \mathfrak{h}'|_{V'_c}}$. Thus the formula (38) shows that, if the jump relations are satisfied under the assumption that H' is compact then they are satisfied in general. Therefore, we can assume that H' is compact.

Fix an element $x' \in \mathfrak{h}'$ which is annihilated by precisely one non-compact imaginary root of \mathfrak{h}' in $\mathfrak{g}'_{\mathbb{C}}$. Then, by Corollary 3.3, the space V' has a direct sum decomposition

$$V' = V'_0 \oplus V'_1 \oplus V'_2 \oplus \cdots \oplus V'_m,$$

such that $[\mathfrak{g}^{x'}|_{V_0}, \mathfrak{g}^{x'}|_{V_0}]$ is isomorphic to $\mathfrak{sl}_2(\mathbb{R})$, and $\mathfrak{g}^{x'}|_{V'_j} = \mathfrak{h}'|_{V'_j}$ for each $j \geq 1$. Let $x \in \mathfrak{g}$ be a semisimple element in the singular support of the distribution $\text{chc}(x' + \cdot)$. Let

$$V = \bigoplus_k V_k$$

be the x -isotypic decomposition of V . Since $\det(x' + x) = 0$, there are j and k such that $\text{eig}(x'|_{V'_j}) \cap \text{eig}(x|_{V_k}) \neq \emptyset$. (Then $\text{eig}(x'|_{V'_j}) = \text{eig}(x|_{V_k}$.) For each j , let \tilde{V}_j be the sum of all the V_k such that $\text{eig}(x'|_{V'_j}) = \text{eig}(x|_{V_k})$. Let $\mathcal{J} \subseteq \{0, 1, 2, \dots, m\}$ be the set such that $\tilde{V}_j \neq 0$. Then

$$\mathcal{W} = \left(\bigoplus_{j \in \mathcal{J}} \ker(x' + x) \cap \text{Hom}(V'_j, \tilde{V}_j) \right) \oplus \ker(x' + x)^\perp.$$

Let $\mathcal{U}' \subseteq \mathfrak{g}^{x'}$ and $\mathcal{U} \subseteq \mathfrak{g}^x$ be slices through x' and x respectively. Then for $x' \in \mathcal{U}'$ and for $x \in \mathcal{U}$,

$$\text{chc}_{\mathcal{W}}(x' + y) = \prod_{j \in \mathcal{J}} \text{chc}_{\ker(x'+x) \cap \text{Hom}(V'_j, \tilde{V}_j)}(x' + y) \text{chc}_{\ker(x'+x)^\perp}(x' + y),$$

where the last factor is a smooth function. Also, by Proposition 3.4, for each $j \in \mathcal{J} \setminus \{0\}$, the function

$$\mathcal{U}' \ni x' \rightarrow \text{chc}_{\text{Hom}(V'_j, \tilde{V}_j) \cap \ker(x'+x)}(x' + \cdot) \in \mathcal{D}'(\mathcal{U})$$

is smooth. Hence, the jump may occur if and only if $0 \in \mathcal{J}$. In that case we are lead to consider the pair

$$\mathbb{G}^{x'}|_{V'_0}, \mathbb{G}^x|_{\tilde{V}_0} \subseteq \text{Sp}(\ker(x' + x) \cap \text{Hom}(V'_0, \tilde{V}_0)). \quad (39)$$

Now we use Corollary 3.3. If $x'|_{V'_0} = 0$ then $x|_{\tilde{V}_0} = 0$ and the pair (39) coincides either with $(O_{p,q}, \text{Sp}_2(\mathbb{R}))$ or with $(\text{Sp}_{2n}(\mathbb{R}), O_{1,2})$.

If $x'|_{V'_0} \neq 0$ then $x|_{\tilde{V}_0} \neq 0$ and there are complex structures $\mathcal{J}' \in \mathfrak{g}'$ and $\mathcal{J} \in \mathfrak{g}$ such that $x' = r'\mathcal{J}'$ and $x = r\mathcal{J}$, for some $r, r' \in \mathbb{R} \setminus \{0\}$. Furthermore, if we view V' and V as complex vector spaces via the actions of \mathcal{J}' and \mathcal{J} , then

$$\begin{aligned} \mathbb{G}^{x'}|_{V'_0} &= U_{1,1}, \quad \mathbb{G}^x|_{\tilde{V}_0} = U_{p,q}, \\ \ker(x' + x) \cap \text{Hom}(V'_0, \tilde{V}_0) &= \text{Hom}_{\mathbb{C}}(V'_0, \tilde{V}_0). \end{aligned}$$

By Theorem 10 of [BP06b], we know that for the dual pairs $(O_{p,q}, \mathrm{Sp}_2(\mathbb{R}))$, $(\mathrm{Sp}_{2n}(\mathbb{R}), O_{1,2})$ and $(U_{p,q}, U_{1,1})$, the jump relations are satisfied. We deduce the result using a partition of unity. \square

4. PROOF OF THEOREM 2

As in the case of Theorem 1, we need to check the boundedness and the jump relations. The boundedness shall be verified in Proposition 4.1 below, where we use the notation of section 1 without comments. The verification of the jump relations is done via the reduction to smaller cases (as in the proof of Theorem 1), lifting to the Lie algebra via the Cayley transform and localization, as in the proof of Proposition 4.1. We leave the details to the reader.

Proposition 4.1. *Let $f \in \mathcal{D}(\tilde{G}_1)$. Then for any $u \in \mathrm{Sym}(\mathfrak{h}'_{\mathbb{C}})$,*

$$\sup_{h' \in \tilde{H}'_{reg}} \left| L(u) \left(\Delta_{\Psi'}(h') \int_{\tilde{G}} \Theta(\check{p}(h')g) f(g) d\mu(g) \right) \right| < \infty.$$

Proof. Let us write $\Theta = \Theta_{V,V'}$ in order to indicate the dependence of the character Θ on the underlying spaces, V and V' . Consider first the case when -1 is not an eigenvalue of any element of $\mathrm{supp}(f)$.

Let $P = (P^+, P^-)$ be a partition of the set $\{1, 2, \dots, n'\}$:

$$\{1, 2, \dots, n'\} = P^+ \cup P^-. \quad (40)$$

For $\epsilon > 0$ let

$$H'(P, \epsilon) = \{h' \in H' \mid |h'_j + 1| > \epsilon \text{ if } j \in P^+, \text{ and } |h'_j + 1| < 2\epsilon \text{ if } j \in P^-\}. \quad (41)$$

Also, let

$$V'^+ = \sum_{j \in P^+} V'_j, \quad V'^- = \sum_{j \in P^-} V'_j. \quad (42)$$

Then, for $\epsilon > 0$ small enough, the sets (41) are non-empty and form an open covering of H' , and the family of functions

$$\Theta_{V,V'^-}(h'g) f(g) \quad (h' \in \tilde{H}'(P, \epsilon))$$

is bounded in $\mathcal{D}(\tilde{G})$. From now on we fix such an ϵ .

Let $\Psi'_+ = \{\alpha \in \Psi' \mid \underline{\alpha} \subseteq P^+\}$. Then for any $v \in \mathrm{Sym}(\mathfrak{h}'_{\mathbb{C}})$, the function

$$L(v) \frac{\Delta_{\Psi'}(h')}{\Delta_{\Psi'_+}(h')} \quad (h' \in H') \quad (43)$$

is bounded. Furthermore,

$$\begin{aligned} \Delta_{\Psi'}(h') \int_{\tilde{G}} \Theta(\check{p}(h')g) f(g) d\mu(g) \\ = \frac{\Delta_{\Psi'}(h')}{\Delta_{\Psi'_+}(h')} \Delta_{\Psi'_+}(h') \int_{\tilde{G}} \Theta_{V,V'_+}(\check{p}(h')g) \left(\Theta_{V,V'_-}(\check{p}(h')g) f(g) \right) d\mu(g). \end{aligned} \quad (44)$$

Thus we may assume that $V'_0 = 0$ (i.e. $P^- = \emptyset$) and consider only the $h' \in \tilde{H}'$ with $|\check{p}(h')_j + 1| > \epsilon$ for all $1 \leq j \leq n'$. Let

$$\check{p}(h') = \tilde{c}(x')\mathfrak{d}, \quad g = \tilde{c}(x)\mathfrak{d}^{-1}.$$

Then

$$\Theta(\check{p}(h')g) = \Theta(\tilde{c}(x')\tilde{c}(x)) = \Theta(h'\mathfrak{d}^{-1})\tilde{chc}(x' + x)\Theta(g\mathfrak{d}). \quad (45)$$

Thus (44) coincides with

$$\frac{\Delta_{\Psi'}(h')}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')} \Theta(\check{p}(h')\mathfrak{d}^{-1}) \pi_{\mathfrak{g}'/\mathfrak{h}'}(x') \int_{\mathfrak{g}} \tilde{chc}(x' + x) \left(\Theta(\tilde{c}(x)) f(\tilde{c}(x)\mathfrak{d}^{-1}) j_{\mathfrak{g}}(x) \right) d\mu(x),$$

where $\pi_{\mathfrak{g}'/\mathfrak{h}'}(x') = \prod_{\alpha \in \Psi'} \alpha(x')$ and $j_{\mathfrak{g}}$ is the Jacobian of the Cayley transform $c : \mathfrak{g} \rightarrow \mathbb{G}$. Since any derivative of the function $\mathbb{R} \ni t \rightarrow \frac{\sin(t)}{t} \in \mathbb{R}$ is bounded, it is easy to check that for any $v \in S(\mathfrak{h}'_{\mathbb{C}})$, the function

$$L(v) \frac{\Delta_{\Psi'}(h')}{\pi_{\mathfrak{g}'/\mathfrak{h}'}(x')} \quad (h' \in \check{\mathbb{H}}') \quad (46)$$

is bounded. Also,

$$|\Theta(\check{p}(h')\mathfrak{d}^{-1})| \leq \left| \frac{1}{\det(1 + p(h'))_{\mathbb{W}}} \right|.$$

Hence, the theorem follows from [BP06a].

Suppose 1 is not an eigenvalue of any element of $\text{supp}(f)$. Then the elements of the support of the function $f(\mathfrak{d}g)$ don't have -1 as an eigenvalue. Furthermore, $\mathfrak{d} \in \check{\mathbb{G}}' \cap \check{\mathbb{G}}$ and $\Delta_{\Psi'}(h') = \Delta_{\Psi'}(h'\mathfrak{d})$. Hence, by the left invariance of the measure μ , the left hand side of (44) is equal to

$$\Delta_{\Psi'}(h'\mathfrak{d}) \int_{\check{\mathbb{G}}} \Theta(\check{p}(h')\mathfrak{d}g) f(\mathfrak{d}g) d\mu(g), \quad (47)$$

and we are in the case considered previously, with $\check{p}(h')$ replaced by $\check{p}(h')\mathfrak{d}$, so the result follows. Thus we need to consider functions f supported in a completely invariant open neighborhood \mathcal{V} of a semisimple element g_0 which has both 1 and -1 as eigenvalues.

If $\mathbb{D} = \mathbb{C}$, then $\check{\mathbb{G}}' \cap \check{\mathbb{G}}$ is a double cover of the unitary group U_1 . Hence, by choosing \mathcal{V} small enough we may translate, as in (47), by an element of $\check{\mathbb{G}}' \cap \check{\mathbb{G}}$ to reduce to the case considered in (44). In order to resolve the general case we proceed as follows.

For a partition P , as in (40), let

$$H'(P) = \{h' \in H'; |h'_j + 1| > \frac{11}{10}|h'_j - 1| \text{ if } j \in P^+, \text{ and } |h'_j - 1| > \frac{11}{10}|h'_j + 1| \text{ if } j \in P^-\}.$$

The sets $H'(P)$ are not empty and form an open covering of H' . Let

$$V = V^+ \oplus V^-$$

be a direct sum decomposition preserved by g_0 and such that -1 is not an eigenvalue of $g_0|_{V^+}$, and 1 is not an eigenvalue of $g_0|_{V^-}$. We may assume that the neighborhood \mathcal{V} is small enough so that the families of functions

$$\begin{aligned} \mathcal{V} \ni g &\rightarrow \Theta_{V^-, V'^+}(h'g) & (h' \in \check{\mathbb{H}}'(P)), \\ \mathcal{V} \ni g &\rightarrow \Theta_{V^+, V'^-}(h'g) & (h' \in \check{\mathbb{H}}'(P)) \end{aligned}$$

are bounded in $C^\infty(\mathcal{V})$. Then the families of functions

$$\begin{aligned} \mathcal{V} \ni g &\rightarrow \frac{\Theta_{V^-, V'^+}(h'g)}{\Theta_{V^-, V'^+}(\mathfrak{d}h'g)} & (h' \in \check{\mathbb{H}}'(P)), \\ \mathcal{V} \ni g &\rightarrow \frac{\Theta_{V^+, V'^-}(h'g)}{\Theta_{V^+, V'^-}(\mathfrak{d}h'g)} & (h' \in \check{\mathbb{H}}'(P)) \end{aligned}$$

are also bounded in $C^\infty(\mathcal{V})$. Furthermore, for $h' \in \check{\mathbb{H}}'(P)$ and $g \in \mathcal{V}$,

$$\begin{aligned} \Theta(h'g) &= \Theta_{V^+, V'^+}(h'g) \Theta_{V^+, V'^-}(\mathfrak{d}h'g) \frac{\Theta_{V^+, V'^-}(h'g)}{\Theta_{V^+, V'^-}(\mathfrak{d}h'g)} \\ &\quad \cdot \Theta_{V^-, V'^-}(h'g) \Theta_{V^-, V'^+}(\mathfrak{d}h'g) \frac{\Theta_{V^-, V'^+}(h'g)}{\Theta_{V^-, V'^+}(\mathfrak{d}h'g)}. \quad (48) \end{aligned}$$

Let

$$h' = (\mathfrak{d}_+ \tilde{c}(x'^+)) \tilde{c}(x'^-), \quad \mathfrak{d}_+ \in \check{\mathbb{G}}'(V'^+), \quad x'^{\pm} \in \mathfrak{h}'|_{V'^{\pm}},$$

and let

$$\begin{aligned} g &= (\mathfrak{d}_+ \tilde{c}(x^+)) \tilde{c}(x^-), \quad \mathfrak{d}_+ \in \check{\mathbb{G}}(V^+), \\ x^{\pm} \in \mathfrak{g} &\text{ are conjugate to elements of } \mathfrak{g}(V^{\pm}) \text{ respectively.} \end{aligned}$$

Then, as in (45),

$$\begin{aligned}
 & \Theta_{V^+, V'^+}(h'g)\Theta_{V^+, V'^-}(\mathfrak{d}h'g) \\
 = & \Theta_{V^+, V'^+}(h'\mathfrak{d}_+^{-1})\widetilde{ch}c_{V^+, V'^+}(x'^+ + x^+)\Theta_{V^+, V'^+}(g\mathfrak{d}_+) \\
 & \Theta_{V^+, V'^-}(h')\widetilde{ch}c_{V^+, V'^-}(x'^- + x^+)\Theta_{V^+, V'^-}(g) \\
 = & \Theta_{V^+, V'^+}(h'\mathfrak{d}_+^{-1})\Theta_{V^+, V'^-}(h')\widetilde{ch}c_{V^+, V'^+}(x' + x^+)\Theta_{V^+, V'^+}(g\mathfrak{d}_+)\Theta_{V^+, V'^-}(g),
 \end{aligned} \tag{49}$$

where $x' = x'^- + x'^+$. Similarly,

$$\begin{aligned}
 \Theta_{V^-, V'^-}(h'g)\Theta_{V^-, V'^+}(\mathfrak{d}h'g) = \Theta_{V^-, V'^-}(h')\Theta_{V^-, V'^+}(h'\mathfrak{d}_+^{-1})\widetilde{ch}c_{V^-, V'^-}(x' + x^-) \\
 \Theta_{V^-, V'^-}(g)\Theta_{V^-, V'^+}(g\mathfrak{d}). \tag{50}
 \end{aligned}$$

Hence, (48) is equal to

$$\begin{aligned}
 & \Theta_{V^+, V'^+}(h'\mathfrak{d}_+^{-1})\Theta_{V^+, V'^-}(h')\Theta_{V^-, V'^-}(h')\Theta_{V^-, V'^+}(h'\mathfrak{d}_+^{-1})\widetilde{ch}c_{V^+, V'^+}(x' + x) \\
 & \left(\Theta_{V^+, V'^+}(g\mathfrak{d}_+)\Theta_{V^+, V'^-}(g)\Theta_{V^-, V'^-}(g)\Theta_{V^-, V'^+}(g\mathfrak{d}^{-1}) \frac{\Theta_{V^+, V'^-}(h'g)}{\Theta_{V^+, V'^-}(\mathfrak{d}h'g)} \frac{\Theta_{V^-, V'^+}(h'g)}{\Theta_{V^-, V'^+}(\mathfrak{d}h'g)} \right). \tag{51}
 \end{aligned}$$

The term in parenthesis is a smooth function and the product of the first four terms on the right hand side of (51) is dominated by

$$\left| \frac{1}{\det(1 + \widetilde{p}(h'))_{\text{Hom}(V^+, V'^+)}} \frac{1}{\det(1 - \widetilde{p}(h'))_{\text{Hom}(V^+, V'^-)}} \frac{1}{\det(1 - \widetilde{p}(h'))_{\text{Hom}(V^-, V'^+)}} \frac{1}{\det(1 + \widetilde{p}(h'))_{\text{Hom}(V^-, V'^+)}} \right|.$$

Thus, since the $\widetilde{ch}c_{V^+, V'^+}$ in (51) corresponds to the situation when the rank of \mathfrak{g}' is less or equal to the rank of \mathfrak{g} the theorem follows from [BP06a]. \square

5. PROOF OF THEOREM 3

As before, let Θ be the distribution character of the Shale Weil (oscillator) representation (of $\widetilde{\text{Sp}}(W)$). In the first subsection, we prove that there exists a certain subset on which we can restrict Θ . This proves that *Chc* has a nice restriction on a dense open subset of \widetilde{G}' denoted $\widetilde{G}'_{\text{npb}}$. The Proposition 5.3 says that the compatibility of *Chc* with the Capelli Harish-Chandra homomorphism is satisfied on $\widetilde{G}'_{\text{npb}}$. The crucial point is that any regular element belonging to a fundamental Cartan subgroup belongs to $\widetilde{G}'_{\text{npb}}$. The extension to the whole \widetilde{G} is immediate from Theorem 2. In the second subsection, we introduce some notations and recall some classical results due to Harish-Chandra. In the last subsection, we prove Theorem 3.

5.1. The restriction of Θ to a dense subset of $\widetilde{G}\widetilde{G}'$. For $g \in \text{Sp}(W)$ the tangent space $T_g\text{Sp}(W)$ may be identified with $g\mathfrak{sp}(W) \subseteq \text{End}(W)$. Then the dual space $T_g^*\text{Sp}(W)$ is identified with $\{g\} \times \mathfrak{sp}^*(W)$ by

$$\mathfrak{sp}^*(W) \ni \xi \rightarrow (T_g\text{Sp}(W) \ni gx \rightarrow \xi(x) \in \mathbb{R}) \in T_g^*\text{Sp}(W)$$

and the cotangent bundle $T^*\mathfrak{sp}(W)$ with $\text{Sp}(W) \times \mathfrak{sp}^*(W)$. Denote by ρ the canonical projection

$$\rho : T^*\mathfrak{sp}(W) \ni (g, \xi) \rightarrow g \in \text{Sp}(W).$$

Similarly, $T^*\widetilde{\text{Sp}}(W)$ is identified with $\widetilde{\text{Sp}}(W) \times \mathfrak{sp}^*(W)$.

Let (G, G') be a dual pair in $\text{Sp}(W)$. The conormal bundle to the embedding $\widetilde{G}\widetilde{G}' \rightarrow \widetilde{\text{Sp}}(W)$ is given by

$$N_{\widetilde{G}\widetilde{G}'} = \{(gg', \xi) \mid g \in \widetilde{G}, g' \in \widetilde{G}', \xi \in \mathfrak{sp}^*(W), \xi|_{\mathfrak{g}+\mathfrak{g}'} = 0\}.$$

Recall, [Prz00, Lemma 12.2], the wave front set of the distribution character $\Theta \in \mathcal{D}'(\widetilde{\text{Sp}}(W))$ of the oscillator representation is

$$\text{WF}(\Theta) = \{(g, \tau_{\mathfrak{sp}}(w)) \mid g \in \widetilde{\text{Sp}}(W) \times \mathfrak{sp}^*(W), w \in W, w \neq 0, \underline{g}(w) = w\}, \quad (52)$$

where $\tau_{\mathfrak{g}} : W \rightarrow \mathfrak{g}^*$, $\tau(w)(x) = \langle xw, w \rangle$. Also, the elements of $\widetilde{\text{Sp}}(W)$ act on W via the map (1). Clearly,

$$\begin{aligned} & \text{WF}(\Theta) \cap N_{\widetilde{\text{G}}\widetilde{\text{G}}'} \\ &= \{(p, \tau_{\mathfrak{sp}}(w)) \in \widetilde{\text{G}}\widetilde{\text{G}}' \times \mathfrak{sp}(W) \mid w \in W \setminus \{0\}, \tau_{\mathfrak{g}}(w) = 0, \tau_{\mathfrak{g}'}(w) = 0, pw = w\}, \end{aligned} \quad (53)$$

and therefore,

$$\rho(\text{WF}(\Theta) \cap N_{\widetilde{\text{G}}\widetilde{\text{G}}'}) = \{p \in \widetilde{\text{G}}\widetilde{\text{G}}' \mid w \in W \setminus \{0\} \text{ with } \tau_{\mathfrak{g}}(w) = 0, \tau_{\mathfrak{g}'}(w) = 0, pw = w\}.$$

Thus

$$\widetilde{\text{G}}\widetilde{\text{G}}' \setminus (\rho(\text{WF}(\Theta) \cap N_{\widetilde{\text{G}}\widetilde{\text{G}}'})) = \{p \in \widetilde{\text{G}}\widetilde{\text{G}}' \mid \tau_{\mathfrak{g}}^{-1}(0) \cap \tau_{\mathfrak{g}'}^{-1}(0) \cap \ker(p - \text{id}_W) = (0)\}.$$

Let $\widetilde{\text{G}}'_{\leq n}$ the preimage of the set of all elements of G' which do not preserve any nonzero (isotropic in the type I case) subspaces of the defining module for \mathfrak{g}' , of dimension less or equal to n .

Lemma 5.1. *Let V denote the defining module for G . If G is the isometry group of a form $(,)$ on V , denote by n the Witt index of the form $(,)$. If $G = \text{GL}(V)$, let $n = \dim(V)$. Then the set of the elements $g' \in \widetilde{\text{G}}'$, such that $gg' \in \widetilde{\text{G}}\widetilde{\text{G}}' \setminus \rho(\text{WF}(\Theta) \cap N_{\widetilde{\text{G}}\widetilde{\text{G}}'})$ for all $g \in \widetilde{\text{G}}$, is equal to $\widetilde{\text{G}}'_{\leq n}$.*

Proof. Suppose the pair (G, G') is of type I. Then G' is the Lie algebra of the isometries of a form $(,)'$ on V' , and the symplectic space W can be realized as $W = \text{Hom}(V, V')$. For $w \in W$ define $w^* \in \text{Hom}(V', V)$ by

$$(w(v), v')' = (v, w^*(v')) \quad (v \in V, v' \in V').$$

Let $g \in \widetilde{\text{G}}, g' \in \widetilde{\text{G}}'$ and $w \in W \setminus \{0\}$ be such that $\tau_{\mathfrak{g}}(w) = 0$, $\tau_{\mathfrak{g}'}(w) = 0$ and $gg'(w) = w$. Then $w^*w = 0$ and $ww^* = 0$. Hence, $\text{im}(w) \subseteq \ker(w^*) = \text{im}(w)^\perp$, and $\ker(w) \supseteq \text{im}(w^*) = \ker(w)^\perp$. Thus $\text{im}(w) \subseteq V'$ is an isotropic subspace, and $\ker(w) \subseteq V$ is a co-isotropic subspace. In particular the dimension of the image of w is not greater than the Witt index of the form $(,)'$. The equation $gg'(w) = w$ translates to $g'w = wg$ and implies that g' preserves $\text{im}(w)$.

Conversely, suppose $g' \in G'$ preserves an isotropic subspace $X' \subseteq V'$ of dimension not greater than the Witt index of $(,)'$. Then there is an isotropic subspace $X \subseteq V$ and a linear bijection $w : X \rightarrow X'$. Define an element $g \in \text{GL}(X)$ by

$$g'w = wg.$$

Let $Y \subseteq V$ be a subspace complementary to X^\perp , and let $U = (X + Y)^\perp$. Then Y is isotropic, $V = X \oplus U \oplus Y$ and g extends to an element $g \in G$ which preserves X, Y and U . Let us extend w to an element of W so that $\ker(w) = U \oplus Y (= X^\perp)$. Then w is a non-zero element of W with $w^*w = 0$, $ww^* = 0$ and $g'w = wg$.

Suppose, from now on, that the pair (G, G') is of type II. Let us realize the symplectic space W as $W = \text{Hom}(V, V') \oplus \text{Hom}(V', V)$. Let $g \in G, g' \in G'$ and $w \in W \setminus \{0\}$ be such that $\tau_{\mathfrak{g}}(w) = 0$, $\tau_{\mathfrak{g}'}(w) = 0$ and $gg'(w) = w$. Then there are $S \in \text{Hom}(V, V')$ and $T \in \text{Hom}(V', V)$, with S or T non-zero, such $w = (S, T)$. Hence our condition translates to, $ST = 0$, $TS = 0$, $g'Sg^{-1} = S$, and $gTg'^{-1} = T$. Clearly, g' preserves the image of S and the kernel of T . Furthermore, $\dim(\text{im}(S)) \leq \dim(V)$, $\text{co} - \dim(\ker(T)) \leq \dim(V)$ and at least one of these spaces is not zero.

Conversely, suppose $g' \in G'$ preserves a non-zero subspace $X' \subseteq V'$, with $\dim(X') \leq \dim(V)$. Let $X \subseteq V$ be a subspace of the same dimension. Let $Y \subseteq V$ be a complementary subspace. Let $S \in \text{Hom}(V, V')$ be such that $\ker(S) = Y$ and $S|_X : X \rightarrow X'$ is a bijection. There is $g \in G$ with $g'Sg^{-1} = S$. Let $T = 0$, and let $w = (S, T)$. Then $w \neq 0$ and $ST = 0$, $TS = 0$, $g'Sg^{-1} = S$, and $gTg'^{-1} = T$.

□

Let $m : \tilde{G} \times \tilde{G}' \ni (g, g') \rightarrow gg' \in \widetilde{\text{Sp}}(W)$ denote the multiplication. Lemma 5.1 implies that the pull-back $m^*(\Theta) \in \mathcal{D}'(\tilde{G} \times \tilde{G}'_{\leq n})$ is well defined. Let K_Θ denote the corresponding integral kernel operator:

$$\begin{aligned} K_\Theta &: D(\tilde{G}) \rightarrow \mathcal{D}'(\tilde{G}'_{\leq n}), \\ K_\Theta(\psi)(\psi') &= m^*(\Theta)(\psi \otimes \psi') \quad (\psi \in \mathcal{D}(\tilde{G}), \psi' \in \mathcal{D}(\tilde{G}'_{\leq n})). \end{aligned}$$

Let $G'_{\text{npb}} \subseteq G'$ be the set of all the elements of G' which do not belong to any proper parabolic subgroup of G' . In other words $G'_{\text{npb}} = G'_{\leq \infty}$ is the set of all the elements of the group which do not preserve any non-zero (isotropic in the type I case) subspaces of V' . In particular $G'_{\text{npb}} \subseteq G'_{\leq n}$ for any $n = 1, 2, \dots$. Clearly it might happen that the set G'_{npb} is empty. In fact G'_{npb} coincides with the set of the orbits passing through the regular elements of a compact Cartan subgroup of G' . In the following lemma we shall focus on the cases when $G'_{\text{npb}} \neq \emptyset$.

Lemma 5.2. *For any $\psi \in \mathcal{D}(\tilde{G})$, the restriction of the distribution $K_\Theta(\psi)$ to \tilde{G}'_{npb} is a smooth function. We shall denote this function by*

$$\int_{\tilde{G}} \Theta(g'g) \psi(g) d\mu(g) \quad (g' \in \tilde{G}'_{\text{npb}}).$$

Proof. We know from [Hör83, Theorem 8.2.12] and (52) that

$$\begin{aligned} \text{WF}(K_\Theta(\psi)|_{\tilde{G}'_{\text{npb}}}) &\subseteq \{(g', \tau_{g'}(w)) \mid g' \in \tilde{G}'_{\text{npb}} \text{ and there is } g \in \text{supp}(\psi) \\ &\text{and } w \in W \setminus \{0\}, \text{ such that } \tau_{g'}(w) = 0 \text{ and } g'g(w) = w\}. \end{aligned}$$

Let g, g' and w be as above. Then

$$\tau_{g'}(w) = \tau_{g'}(gg'(w)) = g' \tau_{g'}(w) g'^{-1}. \quad (54)$$

On the other hand, $\tau_{g'}(w) = 0$ implies that $\tau_{g'}(w)$ is nilpotent. But since $g' \in \tilde{G}'_{\text{npb}}$, (54) implies that this nilpotent element has to be zero, (see [CM93, Lemma 3.8.4]). Therefore $\text{WF}(K_\Theta(\psi)|_{\tilde{G}'_{\text{npb}}}) = \emptyset$, and the lemma follows. \square

Recall the left regular representation of \mathfrak{g} :

$$L(x)\psi(g) = \frac{d}{dt} \psi(\exp(-tx)g)|_{t=0} \quad (x \in \mathfrak{g}, g \in \tilde{G}, \psi \in C^\infty(\tilde{G})), \quad (55)$$

and similarly for \mathfrak{g}' . Then L extends to an injective homomorphism from the universal enveloping algebra $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ to the algebra of the analytic differential operators on \tilde{G} . Recall the involution $\mathcal{U}(\mathfrak{g}_{\mathbb{C}}) \ni z \rightarrow \check{z} \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ defined by $\check{z} = -z$, $z \in \mathfrak{g}$. The action of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ on the space $\mathcal{D}'(\tilde{G})$, of the distributions, is defined by

$$L(z)u(\psi) = u(L(\check{z})\psi) \quad (u \in \mathcal{D}'(\tilde{G}), z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})).$$

(Notice that this definition is consistent with the embedding of $C^\infty(\tilde{G})$ into $\mathcal{D}'(\tilde{G})$ via the Haar measure.)

Assume from now on that the rank of \mathfrak{g}' is less or equal to the rank of \mathfrak{g} . Recall the Capelli Harish-Chandra homomorphism $\mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G \rightarrow \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}$, [Prz04, equality (5.5)].

Theorem 5.3. *For any $\psi \in \mathcal{D}(\tilde{G})$ the function*

$$\tilde{G}'_{\text{npb}} \ni g' \rightarrow \int_{\tilde{G}} \Theta(g'g)\psi(g) dg \in \mathbb{C}$$

is smooth and invariant under the conjugation by \tilde{G}' . Moreover for $z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G$,

$$\int_{\tilde{G}} \Theta(g'g)L(z)\psi(g) d\mu(g) = L(\mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(z)) \int_{\tilde{G}} \Theta(g'g)\psi(g) d\mu(g) \quad (g' \in \tilde{G}'_{\text{npb}}).$$

The above statements hold with the Θ replaced by Chc (see (8)).

Proof. The first part is clear from Lemma 5.2. Furthermore, in terms of the Weyl calculus as in [Prz00],

$$\begin{aligned} \int_{\tilde{G}} \Theta(g'g)L(z)\psi(g) d\mu(g) &= T(g')\natural T(L(z)\psi)(0) = T(g')\natural T(z)\natural T(\psi)(0) \\ &= T(g')\natural T(\mathcal{C}_{\mathfrak{g},g'}(z))\natural T(\psi)(0) = T(\mathcal{C}_{\mathfrak{g},g'}(z))\natural T(g')\natural T(\psi)(0) \\ &= L(\mathcal{C}_{\mathfrak{g},g'}(\check{z}))(T(g')\natural T(\psi)(0)) = L(\mathcal{C}_{\mathfrak{g},g'}(\check{z})) \int_{\tilde{G}} \Theta(g'g)\psi(g) d\mu(g), \end{aligned}$$

where the third equality follows from [Prz04, 6.12]. \square

5.2. Recapitulation of some results of Harish-Chandra. Let H be a Cartan subgroup of a real reductive group G . Let \mathfrak{h} and \mathfrak{g} denote the Lie algebras of H and G respectively. Fix a positive root system for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$

For a root α let $\mathfrak{g}_{\alpha} \subseteq \mathfrak{g}_{\mathbb{C}}$ denote the corresponding $\mathfrak{h}_{\mathbb{C}}$ -eigenspace. Let $\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \subseteq \mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ denote the center of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$.

Theorem 5.4. [HC51, Lemma 36], [HC56, Lemma 18] *For each element $z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$, there is a unique element $\gamma'(z) \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$ such that $z - \gamma'(z) \in \sum_{\alpha > 0} \mathcal{U}(\mathfrak{g}_{\mathbb{C}})\mathfrak{g}_{\alpha}$. Moreover the map*

$$\mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \ni z \rightarrow \gamma'(z) \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$$

is an injective algebra homomorphism.

Let W be the Weyl group for the pair $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h}_{\mathbb{C}})$. Let $\rho = \frac{1}{2} \sum_{\alpha > 0} \alpha$. Let λ denote the following automorphism of $\mathcal{U}(\mathfrak{h}_{\mathbb{C}})$

$$\lambda(x) = x - \rho(x) \quad (x \in \mathfrak{h}_{\mathbb{C}}).$$

Define

$$\gamma = \lambda \circ \gamma'.$$

Theorem 5.5. [HC56, Lemma 19, Lemma 20] *The map $\gamma : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathcal{U}(\mathfrak{h}_{\mathbb{C}})^W$ is a surjective algebra isomorphism. This map does not depend on the choice of the positive root system. If σ is an automorphism of $\mathfrak{g}_{\mathbb{C}}$ which preserves $\mathfrak{h}_{\mathbb{C}}$, then*

$$\gamma(\sigma(z)) = \sigma(\gamma(z)) \quad (z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})).$$

If $z \rightarrow \check{z}$ denote the anti-automorphism of $\mathcal{U}(\mathfrak{g}_{\mathbb{C}})$ defined by $\check{z} = -z$ if $z \in \mathfrak{g}_{\mathbb{C}}$, then

$$\gamma(\check{z}) = \gamma(\check{z}) \quad (z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})).$$

For a root α , let η_{α} denote the corresponding character of H :

$$(\exp(x))^{\alpha} = \exp(\alpha(x)) \quad (x \in \mathfrak{h}).$$

(We assume that these characters exist.) Set

$$\Delta'(h) = \prod_{\alpha > 0} (1 - h^{-\alpha}) \quad (h \in H).$$

Define $\mathcal{S}_c(H^{reg})$ to be the space of all smooth functions f on H^{reg} , such that the support of f is bounded and for every $z \in \mathcal{U}(\mathfrak{h}_{\mathbb{C}})$,

$$\sup_{h \in H^{reg}} |L(z)f(h)| < \infty. \quad (56)$$

Define a topology on $\mathcal{S}_c(H^{reg})$ by taking all the quantities (56) as seminorms. Let

$$\tilde{T}_G(f)(h) = \Delta'(h) \int_{G/H} f(ghg^{-1}) d\mu(gH) \quad (h \in H^{reg}, f \in \mathcal{D}(G)). \quad (57)$$

Theorem 5.6. [HC57, Theorems 2 and 3] *For a function $f \in \mathcal{D}(G)$, $\tilde{I}_G(f) \in \mathcal{S}_c(\mathbb{H}^{reg})$. Moreover the map*

$$\mathcal{D}(G) \ni f \rightarrow \tilde{I}^G(f) \in \mathcal{S}_c(\mathbb{H}^{reg})$$

is continuous. Furthermore

$$\tilde{I}^G(z.f) = \gamma'(z).\tilde{I}^G(f) \quad (z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}), f \in \mathcal{D}(G)).$$

(Here we identify the z with $L(z)$.)

For $h \in \mathbb{H}$ let

$$\begin{aligned} \Delta(h) &= h^\rho \Delta'(h), \\ \Delta'_{\mathbb{R}}(h) &= \prod_{\alpha > 0, \alpha \text{ real}} (1 - \eta_\alpha(h^{-1})), \\ \epsilon_{\mathbb{R}}(h) &= \text{sign}(\Delta'_{\mathbb{R}}(h)) \end{aligned} \tag{58}$$

Theorem 5.7. [HC65, Sec. 22] *For a function $f \in \mathcal{D}(G)$, and for $h \in \mathbb{H}^{reg}$ define*

$$\check{I}^G(f)(h) = \epsilon_{\mathbb{R}}(h)\Delta(h) \int_{G/\mathbb{H}} f(g.h) dg$$

Then

$$\check{I}^G(z.f) = \gamma(z).\check{I}^G(f) \quad (z \in \mathcal{Z}, f \in \mathcal{D}(G)).$$

Remark. *In the introduction, we defined an unnormalized orbital integral denoted I^G which is not bounded. The normalized orbital integrals \tilde{I}^G and \check{I}^G are bounded. The last one is the original Harish-Chandra orbital integral.*

If needed we shall write $\epsilon_{\mathbb{R}}^G$ and Δ^G , in order to indicate the group with respect to which these functions are defined. Consider a parabolic subalgebra $\mathfrak{q} \subset \mathfrak{g}$, with the Langlands decomposition

$$\mathfrak{q} = \mathfrak{m} \oplus \mathfrak{a} \oplus \mathfrak{n}.$$

From now on we assume that the Cartan subalgebra \mathfrak{h} is θ -stable, and that $\mathfrak{h} \cap \mathfrak{p} = \mathfrak{a}$. Let $\mathcal{Z}(\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}})$ denote the center of the universal enveloping algebra of the complexification of $\mathfrak{m} \oplus \mathfrak{a}$. As in [HC65, Section 12] set

$$\nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} = \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}} : \mathcal{Z}(\mathfrak{g}_{\mathbb{C}}) \rightarrow \mathcal{Z}(\mathfrak{m}_{\mathbb{C}} \oplus \mathfrak{a}_{\mathbb{C}}). \tag{59}$$

Let Q be the parabolic subgroup of G such that $\text{Lie}(Q) = \mathfrak{m} \oplus \mathfrak{a}$ and L the Levi factor of Q . We have $\text{Lie}(L) = \mathfrak{m} \oplus \mathfrak{a}$. For $f \in \mathcal{D}(G)$, denote by f^Q the Harish-Chandra transform of f (see (0.3) of [BP06b]).

Theorem 5.8. [HC66, Corollary 2, page 94, and Corollary of Lemma 14, page 96] *For any function $f \in \mathcal{D}(G)$, and any $z \in \mathcal{Z}(\mathfrak{g}_{\mathbb{C}})$*

$$(z.f)^L = \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}(z)f^L.$$

Moreover,

$$\check{I}^G(f) = \check{I}^L(f^L) \quad (f \in \mathcal{D}(G)).$$

Recall, [HC51, p. 117], the Harish-Chandra radial component map $\delta'_{G/\mathbb{H}}$ from the algebra of analytic differential operators on G to the algebra of analytic differential operators on \mathbb{H}^{reg} . For any analytic $\text{Ad}(G)$ -invariant differential operator D on G , $\delta'_{G/\mathbb{H}}(D)$ is the unique analytic differential operator on \mathbb{H}^{reg} such that

$$D\psi|_{\mathbb{H}^{reg}} = \delta'_{G/\mathbb{H}}(D)(\psi|_{\mathbb{H}^{reg}}) \quad (\psi \in C^\infty(G.[\mathbb{H}^{reg}]^G),$$

(see [Var77, Proposition 6, p. 225]). Let $\pi_{G/\mathbb{H}}$ denote any analytic square root of the determinant

$$\mathbb{H}^{reg} \ni h \rightarrow \det(\text{Ad}(h^{-1}) - 1)_{\mathfrak{g}/\mathfrak{h}} \in \mathbb{C}.$$

Then

$$\delta'_{G/\mathbb{H}}(L(z)) = \pi_{G/\mathbb{H}}^{-1} L(\gamma_{\mathfrak{g}/\mathfrak{h}}(z)) \pi_{G/\mathbb{H}} \quad (z \in \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G), \tag{60}$$

(see [HC56, Theorem 2, p. 125] and [HC65, Lemma 13, p.466]).

5.3. The *Chc* and the Capelli Harish-Chandra Homomorphism. Let $(G, G') \subseteq \mathrm{Sp}(W)$ be a dual pair with the rank of G' less or equal to the rank of G . Recall the Cartan subgroup $H' \subseteq G'$ and the parabolic subgroups $Q' \subseteq G'$, $Q \subseteq G$. We identify \mathfrak{h}' with a subspace of \mathfrak{g} , as in [Prz04, Proposition 1.14]. Let $\mathfrak{z} \subseteq \mathfrak{g}$ be the centralizer of \mathfrak{h}' and let $Z \subseteq G$ be the normalizer of \mathfrak{z} in G . If the pair (G, G') is of type I, then there is a non-degenerate subspace $U^0 \subseteq U$ such that $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}''$, with $\mathfrak{z}'' = \mathfrak{g}(U^0)$. If the pair (G, G') is of type II, then $\mathfrak{z} = \mathfrak{h}' \oplus \mathfrak{z}''$, with $\mathfrak{z}'' = \mathfrak{gl}(U)$. Let $Z'' = G(U^0)$ in the first case, and let $Z'' = \mathrm{GL}(U)$ in the second case. Then

$$\mathcal{U}(\mathfrak{z}_{\mathbb{C}})^Z = \mathcal{U}(\mathfrak{h}'_{\mathbb{C}})^{W'} \otimes \mathcal{U}(\mathfrak{z}''_{\mathbb{C}})^{Z''},$$

where W' is the appropriate Weyl group for G' . Let

$$\epsilon_{\mathfrak{z}''} : \mathcal{U}(\mathfrak{z}''_{\mathbb{C}}) \rightarrow \mathbb{C}$$

be the algebra homomorphism by which $\mathcal{U}(\mathfrak{z}''_{\mathbb{C}})$ acts on the trivial representation of \mathfrak{z}'' . Choose a Cartan subalgebra $\mathfrak{h}'' \subseteq \mathfrak{z}''$. Then $\mathfrak{h} = \mathfrak{h}' \oplus \mathfrak{h}''$ is a Cartan subalgebra of \mathfrak{g} . Recall the Capelli Harish-Chandra homomorphism, [Prz04, equality (5.5)],

$$\mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} : \mathcal{U}(\mathfrak{g}_{\mathbb{C}})^G \longrightarrow \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}, \quad \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} = \gamma_{\mathfrak{g}'/\mathfrak{h}'}^{-1} \circ (1 \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}}. \quad (61)$$

Similarly, in the type I case we have,

$$\mathcal{C}_{\mathfrak{g}(U), \mathfrak{g}(V'_c)} : \mathcal{U}(\mathfrak{g}(U)_{\mathbb{C}})^{G(U)} \rightarrow \mathcal{U}(\mathfrak{g}(V'_c)_{\mathbb{C}})^{G(V'_c)}.$$

Also, let $\mathfrak{h}'_s = \mathfrak{h}'|_{X'} \subseteq \mathfrak{gl}(X')$ and let $\mathfrak{h}'_c = \mathfrak{h}'|_{V'_c} \subseteq \mathfrak{g}(V'_c)$. Then

$$\mathcal{U}(\mathfrak{h}'_{\mathbb{C}})^{W'} = \mathcal{U}(\mathfrak{h}'_{s, \mathbb{C}})^{W'_s} \otimes \mathcal{U}(\mathfrak{h}'_{c, \mathbb{C}})^{W'_c}. \quad (62)$$

where W'_s (resp. W'_c) is the Weyl group of $\mathfrak{h}'_{s, \mathbb{C}}$ (resp. $\mathfrak{h}'_{c, \mathbb{C}}$).

Lemma 5.9. *Suppose the pair (G, G') is of type I. Then*

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} = (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{g}(V'_c)/\mathfrak{h}'_c} \circ \mathcal{C}_{\mathfrak{g}(U), \mathfrak{g}(V'_c)}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}.$$

If the pair (G, G') is of type II, then

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} = (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \epsilon_{\mathfrak{z}''}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}.$$

Proof. From (61) we see that

$$\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} = (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}},$$

and similarly

$$\gamma_{\mathfrak{g}(V'_c)/\mathfrak{h}'_c} \circ \mathcal{C}_{\mathfrak{g}(U), \mathfrak{g}(V'_c)} = (1_c \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}(U)/\mathfrak{h}(U)}^{-1} \circ \gamma_{\mathfrak{g}(U)/\mathfrak{h}(U)},$$

where $\mathfrak{z}(U) = \mathfrak{g}(U) \cap \mathfrak{z} = \mathfrak{h}'_c \oplus \mathfrak{z}''$ and 1_c stands for the identity on $\mathcal{U}(\mathfrak{h}'_{c, \mathbb{C}})^{W'_c}$.

Let us write $1 = 1_s \otimes 1_c$ for the identity, according to the decomposition (62). Then,

$$\begin{aligned} \gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} &= (1 \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{g}/\mathfrak{h}} \\ &= (1_s \otimes (1_c \otimes \epsilon_{\mathfrak{z}''})) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= (1_s \otimes (1_c \otimes \epsilon_{\mathfrak{z}''})) \circ \left(1_s \otimes \gamma_{\mathfrak{z}(U)/\mathfrak{h}(U)}^{-1} \right) \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= \left(1_s \otimes (1_c \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}(U)/\mathfrak{h}(U)}^{-1} \right) \circ (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{g}(U)/\mathfrak{h}(U)}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= \left(\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes (1_c \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}(U)/\mathfrak{h}(U)}^{-1} \right) \circ \gamma_{\mathfrak{g}(U)/\mathfrak{h}(U)} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{g}(V'_c)/\mathfrak{h}'_c} \circ \mathcal{C}_{\mathfrak{g}(U), \mathfrak{g}(V'_c)}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \end{aligned}$$

This verifies the first equality. The second one is simpler:

$$\begin{aligned} \gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'} &= (1 \otimes \epsilon_{\mathfrak{z}''}) \circ \gamma_{\mathfrak{z}/\mathfrak{h}}^{-1} \circ \gamma_{\mathfrak{m} \oplus \mathfrak{a}/\mathfrak{h}} \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= (1 \otimes \epsilon_{\mathfrak{z}''}) \circ (1 \otimes \gamma_{\mathfrak{z}''/\mathfrak{h}''})^{-1} \circ (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \gamma_{\mathfrak{z}''/\mathfrak{h}''}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= (1 \otimes \epsilon_{\mathfrak{z}''}) \circ (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes 1) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}} \\ &= (\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_s} \otimes \epsilon_{\mathfrak{z}''}) \circ \nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}. \end{aligned}$$

□

Proof. [of Theorem 3] Consider first a dual pair (G, G') of type I. Let $\psi \in \mathcal{D}(\tilde{G})$ be supported in a connected set of a single sheet of the covering map $\tilde{G} \rightarrow G$. Our task is to show that for all $x' \in H'^{reg}$,

$$\int_{\tilde{G}} Chc(x'g)L(z)\psi(g) d\mu(g) = \delta'_{G'/H'}(L(\mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(z))) \int_{\tilde{G}} Chc(x'g)\psi(g) d\mu(g). \quad (63)$$

Let A' (resp T') be the split (resp. compact) part of H' . Put, as before,

$$V'_c = \{v \in V' \mid a.v = 0 \forall a \in A'\}$$

Then, there exists a unique complement V'_s of V'_c in V' such that the decomposition

$$V' = V'_c \oplus V'_s$$

is preserved by H' . The space V'_c has a complete polarization

$$V'_s = X' \oplus Y'.$$

Let $x' \in \tilde{H}'^{reg}$. We put $x' = x'_c x'_s$ according to the equality $\tilde{H} = \tilde{T}\tilde{A}$. We can assume that V'_s is contained in V (otherwise Chc is trivial) and we have a decomposition

$$V = V'_s \perp U.$$

Recall that $W_c = \text{Hom}(V'_c, U)$. In Theorem 9 of [BP06b], three cases are considered. The two first ones are simpler than the last one, therefore, we assume that \mathfrak{h}' doesn't act trivially on W_c . We have

$$\begin{aligned} & \epsilon_{\mathbb{R}}^{\text{GL}(X')}(x'_s) \Delta^{\text{GL}(X')}(x'_s) |\det(\text{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det(\text{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'}| \int_{\tilde{G}} Chc_{W_c}(x'g)\psi(g) d\mu(g) \\ &= C_0 \epsilon(x'_s) \int_{\text{GL}(X')/H'_{|X'}} \int_{\tilde{G}(U)} \epsilon_{\mathbb{R}}^{\text{GL}(X')}(x'_s) \Delta^{\text{GL}(X')}(x'_s) \epsilon(\mathfrak{d}x'_s y_s) \\ & \quad Chc_{W_c}(x'_c y_c) \psi^{\tilde{L}}(h.(\mathfrak{d}x'_s)y) d\mu(y) d\mu(hH'_{|X'}). \end{aligned} \quad (64)$$

where the constant C_0 is explicitly known. Since $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{gl}(X') \oplus \mathfrak{g}(U)$, there are finitely many elements $s_i \in \mathcal{U}(\mathfrak{gl}(X')_{\mathbb{C}})^{\text{GL}(X')}$, $c_i \in \mathcal{U}(\mathfrak{g}(U)_{\mathbb{C}})^{\text{G}(U)}$, such that

$$\nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}(z) = \sum_i s_i \otimes c_i.$$

Then, by Theorem 5.8,

$$(L(z)\psi)^{\tilde{L}} = \sum_i (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}.$$

Hence, if we replace ψ by $L(z)\psi$ on the right hand side of (64), we get

$$\begin{aligned} & C_0 \epsilon(x'_s) \sum_i \int_{\text{GL}(X')/H'_{|X'}} \int_{\tilde{G}(U)} \epsilon_{\mathbb{R}}^{\text{GL}(X')}(x'_s) \Delta^{\text{GL}(X')}(x'_s) \epsilon(\mathfrak{d}x'_s y_s) Chc_{W_c}(x'_c y_c) \\ & \quad (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(h.(\mathfrak{d}x'_s)y) d\mu(y) d\mu(hH'_{|X'}). \end{aligned} \quad (65)$$

We may rewrite (65) as follows

$$\sum_i \int_{\tilde{\text{GL}}(X')} \int_{\tilde{G}(U)} \epsilon(\mathfrak{d}x'_s y_s) \tilde{I}_{\mathfrak{d}x'_s}^{\tilde{\text{GL}}(X')}(x) Chc_{W_c}(x'_c y_c) (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(x'y) d\mu(y) d\mu(x). \quad (66)$$

By Theorems 5.3 and 5.7, (66) coincides with

$$\begin{aligned} & C_0 \epsilon(x'_s) \sum_i \int_{\tilde{\text{GL}}(X')} \int_{\tilde{G}(U)} L(\gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_{|X'}}(\check{s}_i)) \left(\epsilon(\mathfrak{d}x'_s y_s) \tilde{I}_{\mathfrak{d}x'_s}^{\tilde{\text{GL}}(X')}(x) \right) \\ & \quad \delta'_{\tilde{G}(V'_c)/\tilde{H}'_c}(L(\mathcal{C}_{\mathfrak{g}(U), \mathfrak{g}(V'_c)}(\check{c}_i))) (Chc_{W_c}(x'_c y_c)) \psi^{\tilde{L}}(x'y) d\mu(y) d\mu(x). \end{aligned} \quad (67)$$

The reason why we have \check{s}_i in (67) rather than s_i , as in (66), is that $s_i \in \mathcal{U}(\mathfrak{g}_C)$ in (66) and $s_i \in \mathcal{U}(\mathfrak{g}'_C)$ in (67). More precisely, on one hand, $\mathrm{GL}(X') \subseteq G'$ acts on the symplectic space $W = \mathrm{Hom}(V, V')$ via the postmultiplication by the inverse and on the other $\mathrm{GL}(X')$ is identified with a subgroup of G via the embedding (37). This inverse forces the transition $\mathfrak{s}_i \rightarrow \check{s}_i$.

The formula (60) implies that (67) is equal to

$$C_0 \left(1 \otimes \frac{1}{\pi_{\tilde{G}(V'_c)/\tilde{H}'_c}} \right) L \left(\sum_i \gamma_{\mathfrak{gl}(X')/\mathfrak{h}'_{X'}}(\check{s}_i) \otimes \gamma_{\mathfrak{g}(V'_c)/\mathfrak{h}'_c} \circ \mathcal{C}_{\mathfrak{g}(U), \mathfrak{g}(V'_c)}(\check{c}_i) \right) \left(1 \otimes \pi_{\tilde{G}(V'_c)/\tilde{H}'_c} \right) \\ \int_{\tilde{\mathrm{GL}}(X')} \int_{\tilde{G}(U)} \tilde{I}_{\mathfrak{d}x'_s}^{\tilde{\mathrm{GL}}(X')}(x) \mathrm{Ch}c_{W_c}(x'_c y_c) \psi^{\tilde{L}}(xy) d\mu(y) d\mu(x). \quad (68)$$

Then, Lemma 5.9 shows that (68) coincides with

$$C_0 \left(1 \otimes \frac{1}{\pi_{\tilde{G}(V'_c)/\tilde{H}'_c}} \right) L \left(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(\check{z}) \right) \left(1 \otimes \pi_{\tilde{G}(V'_c)/\tilde{H}'_c} \right) \\ \int_{\tilde{\mathrm{GL}}(X')} \int_{\tilde{G}(U)} \tilde{I}_{\mathfrak{d}x'_s}^{\tilde{\mathrm{GL}}(X')}(x') \mathrm{Ch}c_{W_c}(x'_c y_c) \psi^{\tilde{L}}(xy) d\mu(y) d\mu(x). \quad (69)$$

Hence, by (64),

$$\left(1 \otimes \frac{1}{\pi_{\tilde{G}(V'_c)/\tilde{H}'_c}} \right) L \left(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(\check{z}) \right) \left(1 \otimes \pi_{\tilde{G}(V'_c)/\tilde{H}'_c} \right) \\ \epsilon_{\mathbb{R}}^{\mathrm{GL}(X')}(\tilde{x}'_s) \Delta^{\mathrm{GL}(X')}(x'_s) |\det(\mathrm{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det(\mathrm{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| \int_{\tilde{G}} \mathrm{Ch}c_W(x'g) \psi(g) d\mu(g) \\ = \epsilon_{\mathbb{R}}^{\mathrm{GL}(X')}(\tilde{x}'_s) \Delta^{\mathrm{GL}(X')}(x'_s) |\det(\mathrm{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det(\mathrm{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| \\ \int_{\tilde{G}} \mathrm{Ch}c_W(x'g) (L(z)\psi(g)) d\mu(g) \quad (70)$$

Notice that

$$\Delta^{\mathrm{GL}(X')}(x'_s) |\det(\mathrm{Ad}(x')_{\mathfrak{n}'})|^{1/2} |\det(\mathrm{Ad}(x'^{-1}) - 1)_{\mathfrak{n}'})| \left(1 \otimes \pi_{\tilde{G}(V'_c)/\tilde{H}'_c} \right) (x') = \Delta^{G'}(x').$$

Hence, (70) may be rewritten as

$$\frac{1}{\epsilon_{\mathbb{R}}^{\mathrm{GL}(X')}(x'_s) \Delta^{G'}(x')} L \left(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(\check{z}) \right) \epsilon_{\mathbb{R}}^{\mathrm{GL}(X')}(x'_s) \Delta^{G'}(x') \int_{\tilde{G}} \mathrm{Ch}c_W(x'g) \psi(g) d\mu(g) \\ = \int_{\tilde{G}} \mathrm{Ch}c_W(x'g) (L(z)\psi(g)) d\mu(g). \quad (71)$$

Since

$$\frac{1}{\epsilon_{\mathbb{R}}^{\mathrm{GL}(X')}(x'_s) \Delta^{G'}(x')} \left(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(\check{z}) \right) \epsilon_{\mathbb{R}}^{\mathrm{GL}(X')}(x'_s) \Delta^{G'}(x') = \delta'_{\tilde{G}'/\tilde{H}'_c}(\mathcal{C}_{\mathfrak{g}, \mathfrak{g}'}(\check{z})),$$

we are done.

We consider now a dual pair (G, G') of type II. Again, we need to verify the equality (63). From Theorem 6.5 of [BP06b], we see that

$$\epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} \mathrm{Ch}c_W(x'g) \psi(g) d\mu(g) \\ = C_0 \int_{G'/H'} \int_{\tilde{\mathrm{GL}}(U)} \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \psi^{\tilde{Q}}(x.(\eta x')u) d\mu(u) d\mu(xH') \quad (72)$$

where C_0 is an explicitly known constant. Since $\mathfrak{m} \oplus \mathfrak{a} = \mathfrak{g}' \oplus \mathfrak{gl}(\mathbb{U})$, there are finitely many elements $s_i \in \mathcal{U}(\mathfrak{g}'_{\mathbb{C}})^{G'}$, $c_i \in \mathcal{U}(\mathfrak{gl}(\mathbb{U})_{\mathbb{C}})^{\text{GL}(\mathbb{U})}$, such that

$$\nu_{\mathfrak{g}/\mathfrak{m} \oplus \mathfrak{a}}(z) = \sum_i s_i \otimes c_i.$$

Then, by Theorem 5.8,

$$(L(z)\psi)^{\tilde{L}} = \sum_i (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}.$$

Hence, if we replace ψ by $L(z)\psi$ on the right hand side of (72), we get

$$C_0 \sum_i \int_{G'/H'} \int_{\tilde{\text{GL}}(\mathbb{U})} \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(x \cdot (\mathfrak{d}x'_s)u) d\mu(u) d\mu(xH'). \quad (73)$$

We may rewrite (61) as follows

$$\sum_i \int_{\tilde{G}'} \int_{\tilde{\text{GL}}(\mathbb{U})} \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \epsilon(\mathfrak{d}x'_s I_{\mathfrak{d}x'_s})(x) (L(s_i) \otimes L(c_i)) \psi^{\tilde{L}}(xu) d\mu(u) d\mu(x). \quad (74)$$

By Theorem 5.7, (74) is equal to

$$\sum_i \int_{\tilde{G}'} \int_{\tilde{\text{GL}}(\mathbb{U})} L(\gamma_{\mathfrak{g}'/\mathfrak{h}'}(\check{s}_i)) \left(\epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') I_{\mathfrak{d}x'}(x) \right) \epsilon_{\mathfrak{z}''}(\check{c}_i) \psi^{\tilde{L}}(xu) d\mu(u) d\mu(x). \quad (75)$$

Lemma 5.9 (ii) shows that (75) coincides with

$$L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g},\mathfrak{g}'}(\check{z})) \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}'} \int_{\tilde{\text{GL}}(\mathbb{U})} I_{\mathfrak{d}x'}(x) \psi^{\tilde{L}}(xu) d\mu(u) d\mu(x). \quad (76)$$

Hence, by (72),

$$\begin{aligned} L(\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g},\mathfrak{g}'}(\check{z})) \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} Chc_{\mathbb{W}}(x'g) \psi(g) d\mu(g) \\ = \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') \int_{\tilde{G}} Chc_{\mathbb{W}}(x'g) (L(z)\psi(g)) d\mu(g). \end{aligned}$$

Since

$$\frac{1}{\epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x')} (\gamma_{\mathfrak{g}'/\mathfrak{h}'} \circ \mathcal{C}_{\mathfrak{g},\mathfrak{g}'}(\check{z})) \epsilon_{\mathbb{R}}^{G'}(x') \Delta^{G'}(x') = \delta'_{\tilde{G}'/\tilde{H}'}(\mathcal{C}_{\mathfrak{g},\mathfrak{g}'}(\check{z})),$$

we are done. \square

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