The Oscillator Representation and Examples of the Cauchy Harish-Chandra Integral

Contents

0.1 Introduction

The purpose of these notes is to shed some light at the oscillator representation, dual pairs, First Fundamental Theorem of the Classical Invariant theory and the notion of Cauchy Harish-Chandra integral. In particular, we recall Howe's Construction of the oscillator representation [H1] and the notion of the wave set of a distribution. We compute the Cauchy Harish-Chandra integral in the case of a small but non-trivial dual pair $(O_2, Sp_2(\mathbf{R}))$.

All we do, except this last computation, may be found in the literature [H1], [H2], [Hö], [P1], [P2].

These notes were prepared by P. Olaya and are based on a seminar lectures given by T. Przebinda in the Fall of 2005.

Chapter 1

A Construction of some Representations

1.1 Fourier Series

Recall the Fourier Series for a square integrable function on a unit circle $S^1 = \{e^{i\theta:\theta\in\mathbf{R}}\}$:

$$f(e^{i\theta}) = \sum_{n\in\mathbf{Z}} f_n \; e^{in\theta}, \qquad f_n = \hat{f}(n) := \int_{S^1} f(e^{i\theta}) e^{-in\theta} \frac{d\theta}{2\pi} \tag{1.1}$$

and the Plancherel formula:

$$\int_{S^1} |f(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{\mathbf{Z}} |f_n|^2 d\theta. \tag{1.2}$$

We shall identify the real vector spaces $\mathbf{R^2}$ with the field of complex numbers $\mathbf{C}$ by

$$\mathbf{R}^2 \ni (x,y) \mapsto z = x + iy \in \mathbf{C}.$$

Let μ denote the usual Lebesgue measure on $\mathbf{R^2}$:

$$\int_{\mathbf{R}^2} f(x,y) dx dy = \int_{\mathbf{C}} f(z) d\mu(z).$$

Then

$$L^2(\mathbf{C}) = \{f : \mathbf{C} \to \mathbf{C}, \quad \| f \|_2^2 = \int_{\mathbf{C}} |f(z)|^2 d\mu(z) < \infty\}.$$

Let r, θ be the polar coordinates in $\mathbf{C}$. Thus, for $f \in L^2(\mathbf{C})$ we have:

$$f(re^{i\theta}) = \sum_{n\in\mathbf{Z}} f_n(r) \; e^{in\theta} \tag{1.3}$$

$$\frac{1}{2\pi} \| f \|_2^2 = \sum_{n\in\mathbf{Z}} \int_{\mathbf{R}^+} |r f_n(r)|^2 \frac{dr}{r}. \tag{1.4}$$

(For r fixed apply (1.1) to $f_r(e^{i\theta}) = f(re^{i\theta})$. Then integrate (1.2) over the group $\mathbf{R}^+$.)

Define:

$$L^2(\mathbf{C})_n = \{g \in L^2(\mathbf{C}) : g(ze^{i\theta}) = g(z)e^{in\theta}, z \in \mathbf{C}\}. \tag{1.5}$$

This is a closed subspace of the Hilbert space $L^2(\mathbf{C})$. The statements (1.3) and (1.4) say that $L^2(\mathbf{C})$ is the Hilbert space sum of the subspaces $L^2(\mathbf{C})_n$:

$$L^2(\mathbf{C}) = \widehat{\bigoplus}_{n\in\mathbf{Z}} L^2(\mathbf{C})_n.$$

1.2 Some operators on $L^2(\mathrm{C})$.

Define the following linear maps $L^2(\mathbf{C}) \to L^2(\mathbf{C})$:

$$\begin{aligned}
\delta_a f(z) &= a^{-1}f(a^{-1}z), & a \in \mathbf{R}^+ \qquad &(1.6)\\
m_b f(z) &= e^{ib|z|^2}f(z), & b \in \mathbf{R} \qquad &(1.7)\\
R_u f(z) &= f(zu), & u \in S^1 \qquad &(1.8)\\
Sf(z) &= f(\bar{z}). & &(1.9)\\
& & &(1.10)
\end{aligned}$$

These operators are unitary. For example:

$$\begin{aligned}
\| \delta_a f \|_2^2 &= \int_{S^1}\int_{\mathbf{R}^+} |a^{-1}rf(ra^{-1}e^{i\theta})|^2 \frac{dr}{r} d\theta & (1.11)\\
&= \int_{\mathbf{R}^+} |a^{-1}rf(a^{-1}re^{i\theta})|^2 \frac{d(ar)}{ar} d\theta = \| f \|_2^2 \,. & (1.12)
\end{aligned}$$

Thus $\delta_a, m_b, R_u, S \in \mathcal{U}(L^2(\mathbf{C}))$, the group of unitary operators on $L^2(\mathbf{C})$.

Fact 1.2.1 *If B is a bounded operator on $L^2(\mathbf{C})_0$ which commutes with all the m_b and all the δ_a, then B coincides with a constant multiple of the identity.*

Proof:

For each $N \in \mathbf{N}$ there is $\varphi_N \in L^2(\mathbf{R})$ s.t.:

$$f_N(r) := \int_{\mathbf{R}} \varphi_N(b)e^{ibr^2} db$$

supp $f_N \subset [0, N+1]$, $f_N \in C^\infty(0,\infty)$, $0 \le f_N(r) \le 1$ and $f(r) = 1$ for $r \in [0, N]$.
Since B commutes with all the multiplications by e^{ibr^2}, it commutes with the multiplication by f_N. Hence

$$B(f_N) = B(f_N f_{N+1}) = f_N B(f_{N+1}).$$

Therefore:

$$B(f_N)(r) = B(f_{N+1})(r), \quad (r < N).$$

Thus the following limit exists:

$$C(r) = \lim_{N\to\infty} B(f_N)(r), \quad (r > 0).$$

Moreover, it is clear that for any compactly supported function $f \in L^2(\mathbf{C})_0$,

$$B(f)(r) = C(r)f(r), \quad (r > 0).$$

(Say *supp* $f \subset [-A, A]$. Then, since $r > 0$,

$$f(r) = \sum_{n \in \mathbf{Z}} f_k e^{2\pi i k(\sqrt{r})^2/2A},$$

and thus

$$B(f)(r) = B(\lim f_N f)(r) = f(r) \lim f_N(r) = C(r)f(r).)$$

Since B is a bounded operator, the function C is bounded. Therefore, by approximation, the last equation holds for all $f \in L^2(\mathbf{C})_0$.
Notice that

$$\begin{aligned}
\delta_a B \delta_{a^{-1}} f(z) &= a^{-1}(B\delta_{a^{-1}} f)(a^{-1}z) \\
&= a^{-1}C(a^{-1}r)(\delta_{a^{-1}} f)(a^{-1}z) \\
&= C(a^{-1}r)f(z).
\end{aligned}$$

However, by our assumption, $\delta_a B \delta_{a^{-1}} = B$. Thus $C(r) = C(a^{-1}r)(a, r > 0)$, and hence C is constant.
$\diamond$

Corollary 1.2.1 *If B is a bounded operator on $L^2(\mathbf{C})_n$, which commutes with all the δ_a, m_b, then B is a constant multiple of the identity.*

Proof:

Define B_0 by

$$\begin{tikzcd}
L^2(\mathbf{C})_0 \ni f(r) \arrow[r] \arrow[d, "B_0"] & g(z) = f(r)e^{in\theta} \in L^2(\mathbf{C})_n \arrow[d] \\
L^2(\mathbf{C})_0 \ni B(g)(re^{in\theta})e^{-in\theta} & B(g) \in L^2(\mathbf{C})_n \arrow[l]
\end{tikzcd}$$

Then B_0 is well defined bounded and commutes with all the δ_a, and m_b. By Fact 4.2.2, there is a constant C such that

$$B_0 f(r) = Cf(r).$$

Hence $Bg(z) = Cg(z)$ $\diamond$.
Directly from the definition of the map S we deduce the following equality:

$$S\ L^2(\mathbf{C})_n = L^2(\mathbf{C})_{-n}. \tag{1.13}$$

Corollary 1.2.2 *If B is a bounded operator on $L^2(\mathbf{C})_n + L^2(\mathbf{C})_{-n}$ which commutes with all the δ_a, m_b, S, and R_u, then B is constant multiple of the identity.*

Proof:
Since B commutes with all the R_u it preserves $L^2(\mathbf{C})_n$ and $L^2(\mathbf{C})_{-n}$. Hence by Corollary 1.2.1 there are constants C_n, C_{-n} such that $B|_{L^2(\mathbf{C})_{\pm n}} = C_{\pm n}I$. But Corollary 1.2.2 implies $C_n = C_{-n}$. $\diamond$

1.3 Some Relations between the Operators $\delta_a,\ m_b, S, R_u$

The following relations are easy to verify:

$$\delta_{a_1}\delta_{a_2} = \delta_{a_1 a_2}, \qquad \delta_1 = I; \tag{1.14}$$
$$m_{b_1}m_{b_2} = m_{b_1+b_2}, \qquad m_0 = I; \tag{1.15}$$
$$R_{u_1}R_{u_2} = R_{u_1 u_2}, \qquad R_1 = I; \tag{1.16}$$
$$S^2 = I; \tag{1.17}$$
$$\delta_a m_b \delta_{a^{-1}} = m_{a^{-2}b}; \tag{1.18}$$
$$S R_u S = R_{\bar{u}}; \tag{1.19}$$
$$\delta_a R_u \delta_{a^{-1}} = R_u, \qquad \delta_a S \delta_{a^{-1}} = S; \tag{1.20}$$
$$m_b R_u m_{-b} = R_u, \qquad m_b S m_{-b} = S. \tag{1.21}$$

1.4 A Fourier Transform

Let

$$\mathcal{F}f(z) = \int_{\mathbf{C}} f(z')e^{-i2\pi Re(z\bar{z'})}d\mu(z') \tag{1.22}$$

Then

$$\mathcal{F}^2 f(z) = f(-z) \tag{1.23}$$
$$\mathcal{F}^4 = I \tag{1.24}$$
$$\mathcal{F}^{-1}f(z) = \int_{\mathbf{C}} f(z')e^{i2\pi Re(z\bar{z'})}d\mu(z'). \tag{1.25}$$

Here are some easy to check properties:

$$\mathcal{F}\delta_a\mathcal{F}^{-1} = \delta_{a^{-1}}; \tag{1.26}$$
$$\mathcal{F}R_u\mathcal{F}^{-1} = R_u; \tag{1.27}$$
$$\mathcal{F}S\mathcal{F}^{-1} = S; \tag{1.28}$$

Hence

$$\mathcal{F}L^2(\mathbf{C})_n = L^2(\mathbf{C})_n. \tag{1.29}$$

1.5 Some Relations between Matrices

Some elementary matrix multiplications verify the following identities:

$$\begin{pmatrix} a_1 & 0 \\ 0 & a_1^{-1} \end{pmatrix}\begin{pmatrix} a_2 & 0 \\ 0 & a_2^{-1} \end{pmatrix} = \begin{pmatrix} a_1a_2 & 0 \\ 0 & a_1^{-1}a_2^{-1} \end{pmatrix} \tag{1.30}$$

$$\begin{pmatrix} 1 & 0 \\ b_1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b_1+b_2 & 1 \end{pmatrix} \tag{1.31}$$

$$\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ a^{-2}b & 1 \end{pmatrix} \tag{1.32}$$

$$\begin{pmatrix} c(\theta_1) & s(\theta_1) \\ -s(\theta_1) & c(\theta_1) \end{pmatrix}\begin{pmatrix} c(\theta_2) & s(\theta_2) \\ -s(\theta_2) & c(\theta_2) \end{pmatrix} = \begin{pmatrix} c(\theta_1+\theta_2) & s(\theta_1+\theta_2) \\ -s(\theta_1+\theta_2) & c(\theta_1+\theta_2) \end{pmatrix} \tag{1.33}$$

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} c(\theta) & s(\theta) \\ -s(\theta) & c(\theta) \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} c(\theta) & -s(\theta) \\ s(\theta) & c(\theta) \end{pmatrix} \tag{1.34}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \tag{1.35}$$

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^2 = -I. \tag{1.36}$$

1.6 The oscillator representation of $O_2 \times SL_2(\mathbf{R})$

Theorem 1.6.1 *i) The operators δ_a, m_b generate a subgroup of $\mathcal{U}(L^2(\mathbf{C}))$ isomorphic to*

$$P^+ = \{\begin{pmatrix} a & 0 \\ b & a^{-1} \end{pmatrix} : a \in \mathbf{R}^+, b \in \mathbf{R}\} \subset SL_2(\mathbf{R})\}$$

ii) The operators $\delta_a, m_b, \mathcal{F}$ generate a subgroup of $\mathcal{U}(L^2(\mathbf{C}))$ isomorphic to $SL_2(\mathbf{R})$.

iii) The operators R_u, S generate a subgroup of $\mathcal{U}(L^2(\mathbf{C}))$ isomorphic to O_2.

Proof:
Part *i)* is obvious from (1.14)-(1.16) and (1.30)-(1.32). Let $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in SL_2(\mathbf{R})$. If $\alpha > 0$ then there are unique $a \in \mathbf{R}^+, b, c \in \mathbf{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

If $\alpha < 0$ then there are unique $a \in \mathbf{R}^+, b, c \in \mathbf{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

If $\alpha = 0$, and $\beta > 0$ then there are unique $a \in \mathbf{R}^+, b, c \in \mathbf{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

If $\alpha = 0, \beta < 0$ then there are unique $a \in \mathbf{R}^+, b, c \in \mathbf{R}$ such that:

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

In order to verify *ii)* it suffices to match

$$\begin{pmatrix} a^{-1} & 0 \\ 0 & a \end{pmatrix} \leftrightarrow \delta_a, \quad \begin{pmatrix} 1 & 0 \\ b & 1 \end{pmatrix} \leftrightarrow m_b, \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \leftrightarrow \quad \mathcal{F}.$$

Similarly

$$\begin{pmatrix} c(\theta) & s(\theta) \\ -s(\theta) & c(\theta) \end{pmatrix} \leftrightarrow R_{e^{i\theta}}, \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \leftrightarrow S, \text{ so } iii) \text{ follows.} \qquad \diamond$$

Denote the corresponding isomorphisms by:

$$\omega : SL_2(\mathbf{R}) \to \mathcal{U}(L^2(\mathbf{C})), \qquad and \qquad \omega : O_2 \to \mathcal{U}(L^2(\mathbf{C})) \tag{1.37}$$

Thus we have the following fact.

Fact 1.6.1
- *$\omega(SL_2(\mathbf{R}))$ commutes with $\omega(O_2)$.*
- *$\omega(P^+)$ acts irreducibly on $L^2(\mathbf{C})_0$,*
- *$\omega(P^+)$ and $\omega(SL_2(\mathbf{R}))$ act irreducibly on each $L^2(\mathbf{C})_n$.*
- *$\omega(P^+)\omega(O_2)$ acts irreducibly on each $L^2(\mathbf{C})_n + L^2(\mathbf{C})_{-n}$.*
- *$\omega(P^+)\omega(SL_2(\mathbf{R}))$ acts irreducibly on each $L^2(\mathbf{C})_n + L^2(\mathbf{C})_{-n}$.*

Thus

$$L^2(\mathbf{C}) = \widehat{\bigoplus}_{n\in\mathbf{N}} (L^2(\mathbf{C})_n + L^2(\mathbf{C})_{-n}) \tag{1.38}$$

is the decomposition into irreducibles under the joint action of O_2 and $Sl_2(\mathbf{R})$. (Here $0 \in \mathbf{N}$.)

Also, the representations of $SL_2(\mathbf{R})$ on $L^2(\mathbf{C})_n$ and $L^2(\mathbf{C})_{-n}$ are isomorphic, via S. Hence for $n \neq 0$:

$$L^2(\mathbf{C})_n + L^2(\mathbf{C})_{-n} = L^2(\mathbf{C})_n \otimes C^2, \tag{1.39}$$

where $SL_2(\mathbf{R})$ acts by $\omega(-) \otimes 1$ while O_2 acts by

$$\begin{pmatrix} c(\theta) & s(\theta) \\ -s(\theta) & c(\theta) \end{pmatrix} \longrightarrow 1 \otimes \begin{pmatrix} e^{in\theta} & 0 \\ 0 & e^{-in\theta} \end{pmatrix}, \qquad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \longrightarrow 1 \otimes \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{1.40}$$

Let π_n be the representation of $SL_2(\mathbf{R})$ on $L^2(\mathbf{C})_n$ and Let π'_n be the representation of O_2 on $\mathbf{C}^2$ as in 1.40. Thus, we have shown the following that the restriction of ω to $SL_2(\mathbf{R}) \times O_2$ has the following decomposition:

$$\omega = \sum_{n\in\mathbf{N}} \pi_n \otimes \pi'_n. \tag{1.41}$$

1.7 $O_2, SL_2(\mathbf{R}) \subset Sp_4(\mathbf{R})$.

Let $W = \mathcal{M}_2(\mathbf{R})$, and $\mathcal{J} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. The following formula defines a symplectic form on W:

$$\langle w, w' \rangle = tr(w'^T \mathcal{J} w), \qquad (w, w' \in W).$$

Let $Sp(W) = Sp_4(\mathbf{R}) \subset End(W)$ denote the group preserving the form $\langle\ ,\ \rangle$. We embed O_2 and $SL_2(\mathbf{R})$ into $Sp(W)$ as follows:

$$g(w) = gw, \quad g'(w) = wg'^{-1} \qquad (w \in W, g \in SL_2(\mathbf{R}), g' \in O_2).$$

This definition is correct because

$$\begin{aligned}
\langle g(w), g(w')\rangle &= tr(w'^T g^T \mathcal{J} g w) = tr(w'^T \mathcal{J} w) \\
\langle g'(w), g'(w')\rangle &= tr(w'(g')^T \mathcal{J} g'(w)) = tr((g'^{-1})^T w'^T \mathcal{J} w g'^{-1}) \\
&= tr(w'^T \mathcal{J} w g'^{-1} (g'^{-1})^T) = tr(w'^T \mathcal{J} w) = \langle w, w'\rangle .
\end{aligned}$$

Furthermore, we have the following fact:

Fact 1.7.1 *The groups $O_2, SL_2(\mathbf{R})$ are mutual centralizers in $Sp(W)$.*

Proof:

Let $E_{ij} \in W$ be the matrix with 1 in the i-th row and the j-th column, and zeros elsewhere. We identify $End(W)$ with $\mathcal{M}_2(\mathbf{R})$ via the basis $E_{11}, E_{12}, E_{21}, E_{22}$.
Then the elements of O_2 correspond to matrices

$$\left(\begin{array}{cc} g & 0 \\ 0 & g \end{array}\right), \qquad\qquad (gg^T = I,\ g \in \mathcal{M}_2(\mathbf{R})).$$

Suppose $S \in End(W)$ commutes with O_2. Then in terms of matrices

$$Sg = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right)\left(\begin{array}{cc} g & 0 \\ 0 & g \end{array}\right) = gS = \left(\begin{array}{cc} g & 0 \\ 0 & g \end{array}\right)\left(\begin{array}{cc} A & B \\ C & D \end{array}\right),$$

Hence

$$Ag = gA, \qquad Bg = gB, \qquad Cg = gC, \qquad Dg = gD.$$

By taking $g = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$ or $g = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}\right)$ we see that

$$A = \alpha I, \qquad B = \beta I, \qquad C = \gamma I, \qquad D = \delta I,$$

for some $\alpha, \beta, \gamma, \delta \in \mathbf{R}$. Thus there is $\sigma \in GL_2(\mathbf{R})$ such that:

$$S(w) = \sigma w.$$

But S preserves the form $\langle\ ,\ \rangle$. Thus $\sigma^T \mathcal{J} \sigma = \mathcal{J}$. This means that $\sigma \in Sp_2(\mathbf{R})$. Hence $Sp(W)^{O_2} = Sp_2(\mathbf{R})$.
Conversely, suppose $S \in Sp(W)^{Sp_2(\mathbf{R})}$. Then

$$S = \left(\begin{array}{cc} A & B \\ C & D \end{array}\right), \qquad g = \left(\begin{array}{cc} g & 0 \\ 0 & g \end{array}\right), \qquad\qquad (g \in Sp_2(\mathbf{R})),$$

and $gs = Sg$ translates to

$$gA = Ag, \qquad gB = Bg, \qquad gC = Cg, \qquad gD = Dg.$$

By taking $g = \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array}\right)$ or $g = \left(\begin{array}{cc} a & 0 \\ 0 & a^{-1} \end{array}\right)$ we see that

$$A = \alpha I, \quad B = \beta I, \quad C = \gamma I, D = \delta I,$$

for some $\alpha, \beta, \gamma, \delta \in \mathbf{R}$. Thus there is $\sigma \in GL_2(\mathbf{R})$ such that:

$$S(w) = w\sigma.$$

But S preserves the form $\langle\ ,\ \rangle$. Thus $\sigma^T \sigma = I$, i.e., $S \in O_2$. We conclude that $Sp(W)^{Sp_2(\mathbf{R})} = O_2$. $\diamond$

Chapter 2

The Oscillator Representation

2.1 Fourier Transform in $\mathcal{S}(\mathbf{R}^n)$ and in $\mathcal{S}^*(\mathbf{R}^n)$.

Definition 2.1.1 *$\mathcal{S}(\mathbf{R}^n)$ is the space of all functions $\varphi : \mathbf{R}^n \longrightarrow \mathbf{C}$ such that*

$$p_{\alpha,\beta}(\varphi) := \sup_{x\in\mathbf{R}^n} |x^\beta \partial_x^\alpha \varphi(x)| < \infty$$

for all multi-indices α, β. The topology in $\mathcal{S}(\mathbf{R}^n)$ is defined by the seminorms $p_{\alpha,\beta}$.

For example, any Gaussian

$$e^{-x^T Ax}, \qquad (A = A^T > 0) \tag{2.1}$$

belongs to $\mathcal{S}(\mathbf{R}^n)$. Also, $\mathcal{S}(\mathbf{R}^n)$ is closed under multiplication by polynomials and under taking derivation. Obviously

$$\mathcal{S}(\mathbf{R}^n) \subset L^2(\mathbf{R}^n). \tag{2.2}$$

Define the Fourier transform $\mathcal{F}$, by

$$\mathcal{F}\varphi(\xi) := \int_{\mathbf{R}^n} \varphi(x) e^{-i2\pi x\cdot\xi} dx, \qquad (\xi \in \mathbf{R}^n). \tag{2.3}$$

Theorem 2.1.1 *The Fourier transform $\mathcal{F} : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$ is an isomophism, with inverse given by:*

$$\mathcal{F}^{-1}\varphi(x) := \int_{\mathbf{R}^n} \varphi(\xi) e^{i2\pi x\cdot\xi} d\xi$$

Moreover, $\mathcal{F}^2\varphi(x) = \varphi(-x)$.

We now collect some useful and obvious facts.

Fact 2.1.1 *For each $g \in GL_n(\mathbf{R})$, and $y \in \mathbf{R}^n$, the map $\varphi \to \varphi'$, $\varphi'(x) = \varphi(gx+y)$, is an automorphism of $\mathcal{S}(\mathbf{R}^n)$.*

Fact 2.1.2 *For $\varphi \in \mathcal{S}(\mathbf{R}^n)$,*

- *$\partial_{\xi_i}\mathcal{F}\varphi(\xi) = \mathcal{F}\psi(\xi)$, where $\psi(\xi) = -2\pi i\xi_i\varphi(\xi)$,*
- *$2\pi i\xi_i\mathcal{F}\varphi(\xi) = \mathcal{F}\psi(\xi)$, where $\psi(\xi) = \partial_{\xi_i}\varphi(\xi)$.*

Fact 2.1.3 *Let* $\mathbf{R}^n = \mathbf{R}^p \oplus \mathbf{R}^q$ *and let* $\mathcal{F}_p : \mathcal{S}(\mathbf{R}^p) \to \mathcal{S}(\mathbf{R}^p)$ *denote the Fourier transform, then*

$$\mathcal{F}_p \otimes I : \mathcal{S}(\mathbf{R}^n) \to \mathcal{S}(\mathbf{R}^n)$$

is an isomorphism.

Definition 2.1.2 *The space* $\mathcal{S}^*(\mathbf{R}^n)$*, the dual of* $\mathcal{S}(\mathbf{R}^n)$*, is called the space of tempered distributions on* $\mathbf{R}^n$*.*

The facts 2.1.1-2.1.3 hold in $\mathcal{S}^*(\mathbf{R}^n)$. Also we have the following simple fact.

Fact 2.1.4 *There is an embedding*

$$\mathcal{S}(\mathbf{R}^n) \hookrightarrow \mathcal{S}^*(\mathbf{R}^n), \qquad \varphi(w) \mapsto \varphi(w)dw.$$

2.2 Fourier Transform of a Gaussian

Fact 2.2.1 *(Liouville)*

$$\int_{\mathbf{R}} e^{-\pi x^2} dx = 1.$$

Corollary 2.2.1

$$\int_{\mathbf{R}+iy} e^{-\pi a z^2} dz = a^{-\frac{1}{2}}, \qquad (\text{ if } Re\ a > 0, y \in \mathbf{R}).$$

Fact 2.2.2 *(Fourier Transform of a Gaussian)*

$$\mathcal{F}(e^{-\pi a x^2})(\xi) = e^{-\pi a^{-1}\xi^2} \int_{\mathbf{R}} e^{-\pi a(x+ia^{-1})^2} dx = a^{-\frac{1}{2}} e^{-\pi a^{-1}\xi^2}.$$

This facts generalizes to the n-dimensional case:

Fact 2.2.3

$$\int_{\mathbf{R}^n} e^{-\pi x^T A x} dx = (\det A)^{-\frac{1}{2}}, \qquad (A = A^T,\ Re\ A > 0).$$

This is because both sides agree if $Im A = 0$ and both sides are holomorphic in the set of the indicated matrices, which is convex.

Fact 2.2.4

$$\int_{\mathbf{R}^n+iy} e^{-\pi z^T A z} dz = (\det A)^{-\frac{1}{2}}, \qquad (A = A^T,\ Re\ A > 0).$$

This is due to the fact that the n-form $e^{-\pi z^T A z} dz$ is closed, so that the integral of $d(e^{-\pi z^T A z} dz)$ on the $(n+1)$-chain

$$I \times \mathbf{R}^n \ni (t, x) \mapsto x + iy \in \mathbf{C}^n$$

vanishes.

Let $A = A^T,\ Re\ A > 0$. Then for $x, y \in \mathbf{R}^n$,

$$\begin{aligned} x^T A x + 2ix^T y &= x^T A x + x^T y A i A^{-1} y + (iA^{-1}y)^T A x \\ &= (x + iA^{-1}y)^T A (x + iA^{-1}y) - y^T A^{-1} y \end{aligned}$$

Hence, for A symmetric and positive, we have:

Fact 2.2.5

$$\mathcal{F}(e^{-\pi x^T A x})(y) = \frac{1}{\sqrt{(\det A)}} e^{-\pi y^T A^{-1} y}. \tag{2.4}$$

Theorem 2.2.1 *Let* $B = B^T = \bar{B}, \det B \neq 0$. *Thus, in the sense of distributions, i.e., in* $\mathcal{S}^*(\mathbf{R}^n)$,

$$\mathcal{F}(e^{\pi x^T i B x})(y) = \frac{e^{\frac{i\pi}{4} sgn(B)}}{\sqrt{|\det B|}} e^{-\pi y^T i B^{-1} y} \tag{2.5}$$

Proof:
Let $\epsilon > 0$ and let $A = \epsilon I - iB$ in 2.4. If $\epsilon \to 0$ the left hand side of 2.4 goes to the left hand side of 2.5. Similarly the exponential function of the right hand side of 2.4 goes to the exponential function function of the right hand side of 2.5. In fact, let $b_1, b_2, ..., b_n$ be the eigenvalues of B, then:

$$\begin{aligned} \sqrt{\det(\epsilon I - iB)} &= \prod_{k=1}^{n} \sqrt{(\epsilon - i b_k)} \\ &\xrightarrow[\epsilon \to 0]{} \prod_{k=1}^{n} \sqrt{|b_k|} e^{i\frac{\pi}{4} sg(-b_k)} \end{aligned}$$

Therefore
$$\frac{1}{\sqrt{\det(\epsilon I - iB)}} \xrightarrow[\epsilon \to 0]{} \frac{1}{\sqrt{|\det B|}} \prod_{k=1}^{n} e^{i\frac{\pi}{4} sg(b_k)}$$
◇

2.3 The Oscillator Representation

Fact 2.3.1
- *(Schwartz Kernel Theorem) The following map is an isomorphism of linear topological spaces:*

$$\mathcal{S}(\mathbf{R}^n \oplus \mathbf{R}^n) \ni K \to T_K \in \hom(\mathcal{S}^*(\mathbf{R}^n), \mathcal{S}(\mathbf{R}^n)), \qquad T_K f(x) = \int_{\mathbf{R}^n} K(x, x') f(x') dx'.$$

- *Composition*

$$T_{K_1} T_{K_2} = T_{K_3}, \qquad \int_{\mathbf{R}^n} K(x, x') K(x', x'') dx'.$$

- *The adjoint* $(T_K)^*$ *equals* T_{K^*}, *with* $K^*(x, x') = \overline{K(x', x)}$.

For a matrix $A = A^T$, $Re\ A > 0$, $A \in \mathcal{M}_{2n,2n}(\mathbf{C})$, let

$$K_A(x, x') = e^{-(x^T, x'^T) A \begin{pmatrix} x \\ x' \end{pmatrix}} \qquad (x, x' \in \mathbf{R}^n) \tag{2.6}$$

Then

$$T_{K_{A_1}} T_{K_{A_2}} = f(A_1, A_2) T_{K_{A_3}}, \qquad A_3 = h(A_1, A_2) \tag{2.7}$$
$$(T_{K_A})^* = T_{K_A}$$

Let

$$\| T_K \| = \sup_{\|f\|_2 = 1} \| T_K f \|_2 . \tag{2.8}$$

Then $\| T_{K_A} \| \leq 1$.
Let $W = \mathbf{R}^n \oplus \mathbf{R}^n$, and let

Definition 2.3.1

- *(Symplectic form)*

$$\langle w, w' \rangle = w'^T \mathcal{J} w, \qquad (\mathcal{J} = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}, w, w' \in W). \tag{2.9}$$

- *(Twisted Convolution)*

$$f_1 \natural f_2(w') = \int_W f_1(w) f_2(w - w') e^{i\pi \langle w, w' \rangle} dw, \qquad (f_1, f_2 \in \mathcal{S}(W)). \tag{2.10}$$

- *(The Weyl Transform)*

$$\rho : \mathcal{S}(W) \to \hom(\mathcal{S}^*(\mathbf{R}^n), \mathcal{S}(\mathbf{R}^n))$$

$$\rho(f) = T_{K_f}, \quad K_f(x, x') = \int_{\mathbf{R}^n} f(x - x' + y) e^{i\pi \langle y, x + x' \rangle} dy \quad (x = (x, 0), y = (0, y) \in W).$$

Fact 2.3.2 *The following formulas hold:*

$$\rho(f_1 \natural f_2) = \rho(f_1)\rho(f_2), \quad \rho(f^*) = \rho(f)^*, \quad f^*(w) = \overline{f(w)}.$$

Let

$$\gamma_{\mathcal{A}}(w) = e^{-\pi w^T \mathcal{A} w} \qquad (\mathcal{A} = \mathcal{A}^T, Re\mathcal{A} > 0, w \in W). \tag{2.11}$$

Then it is easy to check that

$$\rho(\gamma_{\mathcal{A}}) = K_A, \qquad A = f(\mathcal{A}). \tag{2.12}$$

By (2.4)

$$\gamma_{\mathcal{A}_1} \natural \gamma_{\mathcal{A}_2} = \det(\mathcal{A}_1 + \mathcal{A}_2)^{-1/2} \gamma_{\mathcal{A}_3}, \qquad \mathcal{A}_3 = F(\mathcal{A}_1, \mathcal{A}_2). \tag{2.13}$$

Definition 2.3.2 *(Normalization:) Put*

$$\gamma^0_{\mathcal{A}} = \pm \det(\mathcal{A} + \frac{i}{2}\mathcal{J})^{\frac{1}{2}} \gamma_{\mathcal{A}}.$$

Then

$$\gamma^0_{\mathcal{A}_1} \natural \gamma^0_{\mathcal{A}_2} = \gamma^0_{\mathcal{A}_3}, \qquad (\mathcal{A}_3 = k(\mathcal{A}_1, \mathcal{A}_2)), \tag{2.14}$$

$$(\gamma^0_{\mathcal{A}})^* = \gamma_{\overline{\mathcal{A}}^0}.$$

Extension of the Weyl Transform:

$$\rho : \mathcal{S}(W)^* \xrightarrow{ext.\ of\ partial\ Fourier\ T.} \mathcal{S}^*(\mathbf{R}^n \oplus \mathbf{R}^n) \xrightarrow{Schwartz\ Kernel\ Theorem} \hom(\mathcal{S}(\mathbf{R}^n), \mathcal{S}^*(\mathbf{R}^n)). \tag{2.15}$$

Fact 2.3.3 *The space of bounded operators on* $L^2(\mathbf{R}^n)$*,*

$$\mathcal{B}(L^2(\mathbf{R}^n)) \subset \hom(\mathcal{S}(\mathbf{R}^n), \mathcal{S}^*(\mathbf{R}^n)).$$

Fact 2.3.4 *Let* $B = B^T \in \mathcal{M}_{2n,2n}(\mathbf{R})$ *and let* $\mathcal{A}$ *be as in (2.11). Then*

- $\lim_{\mathcal{A} \to iB} \rho(\gamma_{\mathcal{A}}) f = \rho(\gamma_{iB}) f, \qquad f \in \mathcal{S}(\mathbf{R}^n).$
- $\rho(\gamma_{iB})\rho(\gamma_{-iB}) = I.$
- $\| \rho(\gamma_{iB}) \| = 1.$
- $\rho(\gamma_{iB_1})\rho(\gamma_{iB_2}) = \rho(\gamma_{iB_3})$, $B_3 = l(B_1, B_2)$.

Definition 2.3.3 *The metaplectic group $\mathcal{M}p$ is the subgroup of $\mathcal{U}(L^2(\mathbf{R}^n))$ generated by all the operators $\rho(\gamma_{iB}),\ \ B = B^T \in \mathcal{M}_{2n,2n}(\mathbf{R})$.*

Definition 2.3.4 *The Cayley transform $c(x) = \frac{x+1}{x-1}$, is a birational isomorphism*

$$c : \mathfrak{sp}(W) \to Sp(W), \qquad\qquad c^2 = I.$$

Let

$$\gamma_{iB_1} \natural \gamma_{iB_2} = \rho^{-1}(\rho(\gamma_{iB_1})\rho(\gamma_{iB_2})).$$

Corollary 2.3.1
- $\gamma_{-\frac{i}{2}\mathcal{J}c(g_1)} \natural \gamma_{-\frac{i}{2}\mathcal{J}c(g_2)} = \gamma_{-\frac{i}{2}\mathcal{J}c(g_1 g_2)} \qquad (g_1, g_2 \in Sp^c(W).$
- $\gamma_{-\frac{i}{2}\mathcal{J}c(g)} \natural \gamma_{-\frac{i}{2}\mathcal{J}c(g^{-1})} = \delta \qquad\qquad (g \in Sp^c(W)).$
- *$\mathcal{M}p \ni \gamma_{-\frac{i}{2}\mathcal{J}c(g)} \to g \in Sp^c(W)$ is a 2 to 1 map preserving group operations.*

Lemma 2.3.1 *(Group Laws are determined by 3/4 majority). Let G be a group and $U \subset G$ be a subset such that $U = U^{-1}$ and $g_1U \cap g_2U \cap g_3U \cap g_4U \neq \emptyset$ $(g_i \in G)$. Let $\underline{G}$ be a group generated by a copy $\underline{U}$ of U such that*

1. *$(\underline{u})^{-1} = \underline{(u^{-1})}$, $u \in U$,*
2. *$\underline{u}_1\underline{u}_2 = \underline{u}_3$ if $u_1u_2 = u_3$, $(u_i \in U)$.*

Then the map $\underline{u} \to u$ extends to an isomorphism $\underline{G} \to G$.

Proof:
Notice first that $G = U^2$. Indeed, if $g \in G$ then $gU \cap U \neq \emptyset$ implies $g \in UU^{-1} = U^2$. There is a homomorphism

$$j : \underline{G} \to G, \quad j(\underline{u}) = u \ (u \in U).$$

By the above, j is a surjective. Since $\underline{G}$, is generated by $\underline{U}$,
$\underline{G} = \underline{U} \cup \underline{U}^2 \cup \underline{U}^3 \cup ...$
We will show that

$$\underline{G} = \underline{U}^2. \tag{2.16}$$

Since $\underline{U} \subset \underline{U}^2$ by the above argument, this will follow from

$$\underline{U}^3 \subset \underline{U}^2. \tag{2.17}$$

Let $u_1, u_2, u_3 \in U$ and $u_4 \in U \cap u_2^{-1}U \cap u_3U \cap (u_1u_2)^{-1}U$.
Then

$$\underline{u_1}\ \underline{u_2}\ \underline{u_3} = \underline{u_1}\ \underline{u_2}\ \underline{u_4}\ \underline{u_4^{-1}}\ \underline{u_3} = \underline{u_1}\ \underline{u_2u_4}\ \underline{u_4^{-1}u_3}$$

$$\underline{u_1u_{)}2u_4}\ \underline{u_4^{-1}u_3} \in \underline{U}^2$$

This verifies (2.17). We need to show that j is injective. Thus if $u_i \in U, i = 1,2,3,4$ and $u_1u_2 = u_3u_4$ then we need to check that $\underline{u_1}\ \underline{u_2} = \underline{u_3}\ \underline{u_4}$. There is

$$u \in U \cap u_2^{-1}U \cap u_4^{-1}U \cap (u_1u_2)^{-1}U \qquad\qquad (\text{ then } u \in (u_3u_4)^{-1}U).$$

Therefore

$$\begin{aligned}
\underline{u_1}\ \underline{u_2} \quad &= \quad \underline{u_1}\ \underline{u_2}\ \underline{u}\ \underline{u}^{-1} = \underline{u_1}\ \underline{u_2u}\ \underline{u}^{-1} = \underline{u_1(u_2u)}\ \underline{u}^{-1} \\
&= \quad \underline{(u_1u_2)u)}\ \underline{u}^{-1} = \underline{(u_3u_4)u)}\ \underline{u}^{-1} = \underline{u_3(u_4u)}\ \underline{u}^{-1} \\
&= \quad \underline{u_3}\underline{u}_4u\ \underline{u}^{-1} = \underline{u_3}\underline{u}_4\underline{u}\ \underline{u}^{-1} = \underline{u_3}\underline{u}_4 \qquad\qquad \diamond
\end{aligned}$$

Let

$$\widetilde{Sp}^c = \{(g,\xi) : g \in Sp^c, \xi^2 = \det(i(g-1))\}$$

This is a real analytic manifold and the map

$$\widetilde{Sp}^c \ni (g,\xi) \to g \in Sp^c(W) \tag{2.18}$$

is a two-fold cover. Let:

$$\begin{aligned}
\chi_x(w) &= e^{\frac{i}{4}\langle xw,w\rangle} \qquad (x \in \mathfrak{sp}(W), w \in W) && (2.19)\\
\Theta(g,\xi) &= \xi && (2.20)\\
T(g,\xi) &= \Theta(g,\xi)\chi_{c(g)} && (2.21)\\
\widetilde{Sp} &= \text{the unique connected 2-fold covering of } Sp \text{ containing } Sp^c. && (2.22)
\end{aligned}$$

We have verified the statements 1,2,3 and 4 of the following theorem:

Theorem 2.3.1 *The map* $T : \widetilde{Sp}^c \to \mathcal{S}^*(W)$ *extends to a unique injective continuous map*

$$T : \widetilde{Sp} \to \mathcal{S}^*(W).$$

Moreover:

1. $T(\widetilde{g_1})\natural T(\widetilde{g_2}) = T(\widetilde{g_1}\widetilde{g_2}) \qquad (\widetilde{g_j} \in \widetilde{Sp}^c)$,
2. $T(\widetilde{g}^{-1}) = T(\widetilde{g})^{-1} \qquad (\widetilde{g} \in \widetilde{Sp}^c)$,
3. $T(1) = \delta$,
4. $\rho(T(\widetilde{Sp})) = \mathcal{M}p \subset \mathcal{U}(L^2(\mathbf{R}^n))$,
5. $tr(\int_{\widetilde{Sp}} E(\tilde{g})\rho(T(\widetilde{g}))d\widetilde{g}) = \int_{\widetilde{Sp}} \Theta(\widetilde{g})E(\widetilde{g}))d\widetilde{g} \qquad (E \in C_c^\infty(\widetilde{Sp}))$.

Proof of (5):

Suppose supp$E \subset \widetilde{Sp}^c$. Then

$$\int_{\widetilde{Sp}} E(\tilde{g})T(\widetilde{g})d\widetilde{g} \in \mathcal{S}(W).$$

Hence,

$$\begin{aligned}
tr(\int_{\widetilde{Sp}} E(\tilde{g})\rho(T(\widetilde{g}))d\widetilde{g}) &= \int_{\mathbf{R}^n}\int_{\widetilde{Sp}} E(\tilde{g})K_\rho(T(\widetilde{g}))(x,x,)d\widetilde{g}dx\\
&= \int_{\widetilde{Sp}} E(\tilde{g})T(\widetilde{g})(0)d\widetilde{g} = \int_{\widetilde{Sp}} \Theta(\widetilde{g})E(\widetilde{g}))d\widetilde{g}. \quad \diamond
\end{aligned}$$

Chapter 3

An Example

3.1 The Oscillator Representation (continued)

We begin by recalling what we verified, in the previous sections, in a coordinate free form. Let W be a finite dimensional vector space over $\mathbf{R}$ with a non-degenerate symplectic form $\langle\ ,\ \rangle$. Let $Sp \subset End(W)$ denote the corresponding symplectic group with Lie algebra $\mathfrak{sp} \subset End(W)$. Let $\mathcal{J}$ be a positive definite complex structure on W. This means $\mathcal{J} \in \mathfrak{sp}$, $\mathcal{J}^2 = -Id$ and the symmetric bilinear form

$$\langle \mathcal{J}w, w'\rangle \qquad\qquad (w, w' \in W) \tag{3.1}$$

is positive definite. Let dw be the Lebesgue measure on W normalized so that the volume of the unit cube, with respect to the norm defined by (3.1) is 1. Fix a character

$$\chi(r) = e^{2\pi i r} \qquad\qquad (r \in \mathbf{R}). \tag{3.2}$$

Let

$$\chi_z(w) = \chi(\frac{1}{4}\langle zw, w\rangle) \qquad\qquad (z \in \mathfrak{sp}_{\mathbf{C}}, w \in W). \tag{3.3}$$

Fix a complete polarization

$$W = X \oplus Y. \tag{3.4}$$

We restrict the scalar product (3.1) to X and Y, and normalize the Lebesgue measures $dx,\ dy$ so that the volumes of the corresponding unit cubes are 1. Recall the Weyl transform

$$\rho \quad : \quad \mathcal{S}^*(W) \to \hom(\mathcal{S}(X), \mathcal{S}^*(X)), \tag{3.5}$$

$$\rho(f) \quad = \quad T_{K_f}, \quad T_{K_f}u(x) = \int_X K_f(x, x')u(x')dx', \tag{3.6}$$

$$K_f(x, x') \quad = \quad \int_Y f(x - x' + y)\xi(\frac{1}{2}\langle y, x + x'\rangle)dy. \tag{3.7}$$

For two distributions $f_1, f_2 \in \mathcal{S}^*(W)$ define the twisted convolution

$$f_1 \natural f_2 = \rho^{-1}(\rho(f_1)\rho(f_2)) \in \mathcal{S}^*(W), \tag{3.8}$$

whenever the composition of operators makes sense. Recall the Cayley transform

$$c(x) = (x + 1)(x - 1)^{-1} \qquad\qquad (x \in \mathrm{end}(W)). \tag{3.9}$$

Let $Sp^c \subset Sp$ denote the domain of c, and let $\mathfrak{sp}^c \subset \mathfrak{sp}$ also denote the domain of c.

Fact 3.1.1 *The Cayley transform $c(x)$, is a birational isomorphism*

$$c : \mathfrak{sp}^c \to Sp^c,$$

with inverse equal to $c : Sp^c \to \mathfrak{sp}^c$. It extends uniquely to the complexifications $c : \mathfrak{sp}^c_{\mathbf{C}} \to Sp^c_{\mathbf{C}}$, by the same formula (3.9).

Fact 3.1.2 *For any $x \in \mathfrak{sp}$, the formula*

$$\langle xw, w' \rangle \qquad\qquad (w, w' \in W)$$

defines a symmetric bilinear form in W.

Let $\mathfrak{sp}^+_{\mathbf{C}} = \{x + iy : x, y \in \mathfrak{sp}, \langle x\ ,\ \rangle > 0, \det(x + iy - 1) \neq 0\}$, and let $Sp^+_{\mathbf{C}} = c(\mathfrak{sp}^+_{\mathbf{C}}) \subset Sp_{\mathbf{C}}$.

Fact 3.1.3 *The subset $Sp^+_{\mathbf{C}} \subset Sp_{\mathbf{C}}$ is a subsemigroup. The closure of $Sp^+_{\mathbf{C}}$ contains Sp. Moreover $Sp^+_{\mathbf{C}} \subset Sp_{\mathbf{C}}$ and $Sp^+_{\mathbf{C}} \subset Sp^c_{\mathbf{C}}$.*

Definition 3.1.1 *Let*

$$\widetilde{Sp}^c = \{\widetilde{g} = (g, \xi) : \xi^2 = \det i(g - 1), g \in Sp^c\} \tag{3.10}$$

$$\widetilde{Sp_{\mathbf{C}}}^+ = \{\widetilde{g} = (g, \xi) : \xi^2 = \det i(g - 1), g \in Sp^+_{\mathbf{C}}\} \tag{3.11}$$

and let $\widetilde{g} \mapsto g$ be the corresponding covering map. Define

$$\Theta(\widetilde{g}) = \xi, \qquad T(\widetilde{g}) = \Theta(\widetilde{g})\chi_{c(g)} \qquad (\widetilde{g} \in \widetilde{Sp}^c \cup \widetilde{Sp}^+_{\mathbf{C}}). \tag{3.12}$$

Thus

$$T : \widetilde{Sp}^c \cup \widetilde{Sp}^+_{\mathbf{C}} \longrightarrow \mathcal{S}^*(W).$$

Finally, let

$$\omega(\widetilde{g}) = \rho(T(\widetilde{g})), \qquad\qquad (\widetilde{g} \in \widetilde{Sp}^c \cup \widetilde{Sp}^+_{\mathbf{C}}). \tag{3.13}$$

Then we have,

Theorem 3.1.1 *The map T extends to a unique injective continuous map*

a)

$$T : \widetilde{Sp} \hookrightarrow \mathcal{S}^*(W)$$

such that for $\widetilde{g}_1, \widetilde{g}_2, \widetilde{g} \in \widetilde{Sp}$

b)

$$\omega(\widetilde{g}_1)\omega(\widetilde{g}_2) = \omega(\widetilde{g}_1\widetilde{g}_2), \qquad\qquad \omega(\widetilde{g})\omega(\widetilde{g}^{-1}) = 1.$$

In the topology of $\mathcal{S}^*(W)$

c)

$$T(\widetilde{g}) = \lim_{\widetilde{Sp}^+_{\mathbf{C}} \ni \widetilde{p} \longrightarrow 1} T(\widetilde{g}\widetilde{p}) \qquad\qquad (\widetilde{g} \in \widetilde{Sp}).$$

For any $\widetilde{g} \in \widetilde{Sp}$, the map

$$\mathcal{S}(W) \ni \varphi \longrightarrow T(\widetilde{g})\natural\varphi \in \mathcal{S}(W)$$

is continuous. Moreover

d)

$$||\omega(\widetilde{g})|| \leq 1 \qquad\qquad (\widetilde{g} \in \widetilde{Sp_{\mathbf{C}}^{+}}).$$

For a subspace $V \subset \mathfrak{sp}$ define the unnormalized moment map

$$\tau_V : W \longrightarrow V^*, \qquad \tau_V(w)(x) = \langle xw, w\rangle \qquad (x \in V, w \in W). \tag{3.14}$$

Lemma 3.1.1 *Let $G, G' \subset Sp$ be a dual pair with G' compact. Then the map*

$$\mathcal{S}(\mathfrak{g}^*) \ni \psi \longrightarrow \psi \circ \tau_{\mathfrak{g}^*} \in \mathcal{S}(W)$$

is well defined and continuous.

Proof:
The assumption that G' is compact is equivalent to the assumption that there is a positive definite complex structure $\mathcal{J}$ as in (3.1), such that $\mathcal{J} \in \mathfrak{g}$. This implies

$$\tau_{\mathfrak{g}}(w) = 0 \Leftrightarrow w = 0. \tag{3.15}$$

Indeed,

$$\tau_{\mathfrak{g}}(w) = 0 \Leftrightarrow (\langle xw, w\rangle = 0 \quad \text{for all } x \in \mathfrak{g}) \Leftrightarrow \langle \mathcal{J}w, w\rangle = 0$$

because $\mathcal{J} \in \mathfrak{g}$, and $\langle \mathcal{J}-, -\rangle$ is positive definite. Furthermore

$$\tau_{\mathfrak{g}} : W \longrightarrow \mathfrak{g}^*$$

is a quadratic map. Hence, if $\|\|$ is a norm on the real vector space $\mathfrak{g}^*$, then there are constants $0 < A < B < \infty$ such that

$$A \parallel w \parallel^2 \leq \parallel \tau_{\mathfrak{g}}(w) \parallel \leq B \parallel w \parallel^2 \quad (w \in W), \tag{3.16}$$

where $\parallel w \parallel^2 = \langle \mathcal{J}w, w\rangle$. We see from (3.16) that if $\psi : \mathfrak{g}^* \longrightarrow \mathbf{C}$ is rapidly decreasing, thus so is

$$\psi \circ \tau_{\mathfrak{g}} : W \longrightarrow \mathbf{C}.$$

Hence, with some more work, the lemma follows. $\diamond$

Lemma 3.1.2 *Let $G, G' \subset Sp$ be a dual pair with G' compact. Then for any $E \in C_c^\infty(\widetilde{G})$,*

$$T(E) = \int_{\widetilde{G}} E(\widetilde{g})T(\widetilde{g})d\widetilde{g} \in \mathcal{S}(W)$$

and the map

$$C_c^\infty(\widetilde{G}) \ni E \mapsto T(E) \in \mathcal{S}(W)$$

is continuous.

Proof:
Let $U = G \cap Sp^c$, and let $\widetilde{U} \subset \widetilde{G}$ be the preimage of U. Then there are $\widetilde{g}_1, \widetilde{g}_2, ..., \widetilde{g}_n \in \widetilde{G}$ such that

$$G = \bigcup_{i=1}^{n} \widetilde{g}_i \widetilde{U}.$$

Fix $E \in C_c^\infty(\widetilde{G})$. There are functions $E_j \in C_c^\infty(\widetilde{G})$ such that *supp* $E_j \subset \widetilde{g}_j \widetilde{U}$ and $E = \sum_{j=1}^n E_j E$. Hence

$$\begin{aligned} T(E) &= \sum_{j=1}^{n} T(E_jE) = \sum_{j=1}^{n} \int_{\widetilde{U}} (E_jE)(\widetilde{g}_j\widetilde{g})T(\widetilde{g}_j\widetilde{g})d\widetilde{g} \\ &= \sum_{j=1}^{n} T(\widetilde{g}_j)\natural \int_{\widetilde{U}} \Phi_j(\widetilde{g})T(\widetilde{g})d\widetilde{g}, \end{aligned}$$

where $\Phi_j(\widetilde{g}) = (E_jE)(\widetilde{g}_j\widetilde{g})$. Hence, by (???) we may assume that $suppE \subset U$. Then

$$\begin{aligned} T(E) &= \int_{\widetilde{G}} E(\widetilde{g})\Theta(\widetilde{g})\chi_{c(g)}d\widetilde{g} \\ &= \int_{\tilde{\mathfrak{g}}} E(\widetilde{c(x)})\Theta(\widetilde{c(x)})j(x)\chi_x dx \end{aligned}$$

where $j(x)$ is the Jacobian for c. Let

$$\varphi(x) = E(\widetilde{c(x)})\Theta(\widetilde{c(x)})j(x).$$

Then $\varphi \in C_c^\infty(\mathfrak{g})$. Let

$$\psi(\xi) = \int_{\mathfrak{g}} \varphi(x)\chi(\frac{1}{4}\xi(x))dx.$$

Then ψ is a Fourier transform of φ, and therefore $\psi \in \mathcal{S}(\mathfrak{g}^*)$. Moreover

$$T(E) = \psi \circ \tau_{\mathfrak{g}}.$$

Thus Lemma 3.1.2 follows from Lemma 3.1.1. $\diamond$

Lemma 3.1.3 *For $\varphi \in \mathcal{S}(W)$, the operator $\rho(T(\varphi))$ is of trace class and* $\mathrm{tr}(\rho(T(\varphi))) = \varphi(0)$.

Proof:

$$\begin{aligned} \mathrm{tr}(\rho(T(\varphi))) &= \mathrm{tr}(T_{K_\rho(\varphi)}) = \int_X K_\rho(\varphi)(x,x)dx && (3.17) \\ &= \int_X\int_Y \varphi(x-x+y)\chi(\frac{1}{2}\langle y, x+x\rangle)dydx = \int_X\int_Y \varphi(y)\chi(\langle y,x\rangle)dydx && (3.18) \\ &= \int_Y \varphi(y)\delta(y)dy = \varphi(0). \qquad \diamond && (3.19) \end{aligned}$$

Let $\mathcal{H} = L^2(X)$. Then ω is a unitary representation of $\widetilde{Sp}$ on the Hilbert space $\mathcal{H}$. Thus for any subgroup of $\widetilde{Sp}$, ω is a unitary representation of that subgroup on $\mathcal{H}$.

Theorem 3.1.2 *(The First Fundamental Theorem of the Classical Invariant Theory) Let $G.G' \subset Sp$ be a dual pair with G'-compact. Suppose Π' is an irreducible unitary representation of $\widetilde{G'}$ such that $\mathcal{H}_{\Pi'}$, the Π'-isotypic component of $\mathcal{H}$, is not zero. Then $\mathcal{H}_{\Pi'}$ is also isotypic for the action of $\tilde{G}$. Let Π denote the corresponding irreducible unitary representation of $\tilde{G}$, and let $\mathcal{H}_\Pi \subset \mathcal{H}$ be the Π-isotypic component. Then*

$$\mathcal{H}_{\Pi'} = \mathcal{H}_\Pi.$$

Thus $\Pi \to \Pi'$ is a bijection between (some) irreducible unitary representations of $\tilde{G}'$ and (some) irreducible unitary representations of $\tilde{G}$.

Theorem 3.1.3 *Let $\tilde{G}$, $\tilde{G}'$, $\mathcal{H}_\Pi$ and $\mathcal{H}'_\Pi$ as in Theorem 3.1.2. Then, for $E \in C^\infty(\tilde{G})$ the operator*

$$\Pi(E) = \int_{\tilde{G}} E(g)\Pi(g)dg$$

is of trace class and, with $G^+_{\mathbf{C}} = G_{\mathbf{C}} \cap Sp^+_{\mathbf{C}}$,

$$\mathbf{tr}\ \Pi(E) = \lim_{G^+_{\mathbf{C}} \ni p \to 1} \int_{\tilde{G}'} \int_{\tilde{G}} \Theta(gpg')\mathbf{tr}(\Pi'(g'))E(g)dgdg'.$$

Thus Θ_Π, the character of Π, exits and, as a distribution

$$\Theta_\Pi(g) = \lim_{G^+_{\mathbf{C}} \ni p \to 1} \int_{\tilde{G}'} \int_{\tilde{G}'} \Theta(gpg')\Theta_{\Pi'}(g')dg',$$

where $\Theta_{\Pi'}(g') = \mathbf{tr}(\Pi'(g'))$. Here we assume that the total Haar measure of $\tilde{G}'$ is 1.

Proof:
Let $\mathcal{P}_{\Pi'} : \mathcal{H} \to \mathcal{H}_{\Pi'}$ denote the orthogonal projection onto $\mathcal{H}_{\Pi'}$. From the representation theory of compact groups we know that

$$\mathcal{P}_{\Pi'} = \int_{\tilde{G}'} d_{\Pi'}\overline{\Theta_{\Pi'}(g')}\omega(g')dg',$$

where $d_{\Pi'} = \dim \Pi'$. By Lemma 3.1.2, $\omega(E)$ is of trace class. Hence $P_{\Pi'}\omega(E)$ is of trace class. Thus by Theorem 3.1.1,

$$\begin{aligned}
\mathrm{tr}(\mathcal{P}_{\Pi'}\omega(E)) &= \mathrm{tr} \int_{\tilde{G}'} \int_{\tilde{G}} d_{\Pi'}\overline{\Theta_{\Pi'}}(g')E(g)\omega(g'g)dgdg' && (3.20)\\
&= \mathrm{tr} \lim_{p \to 1} \int_{\tilde{G}'} \int_{\tilde{G}} d_{\Pi'}\overline{\Theta_{\Pi'}}(g')E(g)\omega(g'gp)dgdg' && (3.21)\\
&= \lim_{p \to 1} \mathrm{tr} \int_{\tilde{G}'} \int_{\tilde{G}} d_{\Pi'}\overline{\Theta_{\Pi'}}(g')E(g)\omega(g'gp)dgdg' && (3.22)\\
&= \lim_{p \to 1} \int_{\tilde{G}'} \int_{\tilde{G}} d_{\Pi'}\overline{\Theta_{\Pi'}}(g')E(g)T(g'gp)(0)dgdg' && (3.23)\\
&= \lim_{p \to 1} \int_{\tilde{G}'} \int_{\tilde{G}} d_{\Pi'}\overline{\Theta_{\Pi'}}(g')E(g)\Theta(g'gp)dgdg'. && (3.24)
\end{aligned}$$

Since Π has multiplicity $d_{\Pi'}$ in $\mathcal{H}'_\Pi$,

$$\mathrm{tr}(\mathcal{P}_{\Pi'}\omega(E)) = d_{\Pi'}\mathrm{tr}(\Pi(E)),$$

and our formula follows. $\diamond$

Example 3.1.1 $G = Sp_2(\mathbf{R}), G' = O_2$.

Put:

$$W = \mathcal{M}_{2,2}(\mathbf{R}),\ \langle w, w' \rangle = \mathrm{tr}\ w'^T \mathcal{J} w,\ g(w) = gw,\ g'(w) = wg'^{-1} \quad (g \in G,\ g' \in G',\ w \in W).$$

Let

$$H = \{\begin{pmatrix} \cos(t) & \sin(t) \\ -\sin(t) & \cos(t) \end{pmatrix} =: h(t) = h, 0 \le t < 2\pi\}.$$

This is a compact Cartan subgroup de G. We shall also view H as a subgroup of G'. Furthermore, we will identify H with S^1, by

$$h(t) = \cos(t)I + \sin(t)\mathcal{J} \longleftrightarrow e^{it}.$$

Let

$$W^{\mathcal{J}} = \{w \in W : w\mathcal{J} = \mathcal{J}w\}, \ {}^{\mathcal{J}}W = \{w \in W : w\mathcal{J} = -\mathcal{J}w\}$$

Then for $h \in H \subset G$, $h' \in H \subset G'$,

$$\begin{aligned} hh'(w) &= hh'^{-1}w & \qquad (w \in W^{\mathcal{J}}) \\ hh'(w) &= hh'w & \qquad (w \in {}^{\mathcal{J}}W). \end{aligned}$$

Hence,

$$\begin{aligned} \det(i(hh'-1))_W &= \det((hh'-1))_W & (3.25) \\ &= \det((hh'-1))_{W^{\mathcal{J}}} \det((hh'-1))_{{}^{\mathcal{J}}W} & (3.26) \\ &= |hh'^{-1}-1|^2|hh'-1|^2. & (3.27) \end{aligned}$$

Let $z, u \in \mathbf{C}, z^2 = u, |u| = 1$. Then

$$|z-1|^2 = |u-u^{-1}|^2 = |(u-u^{-1})^2| = |(u-\overline{u})^2|$$

$$= -(u-\overline{u})^2 = -(u-u^{-1})^2.$$

Let $h = h(t), h' = h'(t')$. Then, by the computation above

$$\begin{aligned} |hh'^{-1}-1|^2|hh'-1|^2 &= (-(h(t/2)h(-t'/2) - h(-t/2)h(t'/2)))^2 & (3.28) \\ &\quad \times(-(h(t/2)h(t'/2) - h(-t/2)h(-t'/2)))^2 & (3.29) \\ &= ((h(t/2)h(-t'/2) - h(-t/2)h(t'/2)) & (3.30) \\ &\quad \times(h(t/2)h(t'/2) - h(-t/2)h(-t'/2)))^2 & (3.31) \\ &= ((hh'^{-1}-1)h^{-1}(1-hh'))^2. & (3.32) \end{aligned}$$

Thus

$$\Theta(\tilde{h}\tilde{h'})^2 = \left(\frac{h}{(1-hh')(1-hh'^{-1})}\right)^2. \tag{3.33}$$

Hence,

$$\Theta(\tilde{h}\tilde{h'}) = \frac{h}{(1-hh')(1-hh'^{-1})} \qquad (h, h' \in \mathbf{C}, \ |h'| = 1) \tag{3.34}$$

is a holomorphic function of h. Notice that

$$\langle y\mathcal{J}-, -\rangle > 0 \Leftrightarrow y < 0 \qquad (y\mathcal{J} \in \mathfrak{g}).$$

and that

$$c(iy\mathcal{J}) = \frac{1}{y^2-1}\begin{pmatrix} y^2+1 & 2iy \\ -2iy & y^2+1 \end{pmatrix} = \frac{y^2+1-2y}{y^2-1} = \frac{y+1}{y-1} = c(y).$$

Thus $y < 0$ implies $|c(y)| < 1$, which justifies (3.34). Let

$$A = \{g(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a > 0\}.$$

This is the identity component of the split Cartan subgroup of G. As before, we compute:

$$\begin{aligned}\det(i(g(a)h'-1))_W &= \det((g(a)h'-1))_W\\ &= \det((ah'-1))_{\mathbf{R}^2}\det((a^{-1}h'-1))_{\mathbf{R}^2}\\ &= |ah'^{-1}-1|^2|a^{-1}h'-1|^2.\end{aligned}$$

Notice that

$$|a^{-1}h'-1| = |h'-a|a^{-1} = |\overline{h'}-a|a^{-1} = |h'^{-1}-a|a^{-1} = |1-ah'|a^{-1}.$$

Hence,

$$\Theta(\widetilde{g(a)}\widetilde{h'}) = |1-ah'|^{-2}|1-ah'|^{-2}a^2$$

and therefore,

$$\Theta(g(a)h') = \frac{a}{|1-ah'|^2} = \frac{a}{|1-ah'||1-ah'^{-1}|} \qquad (a>0, h'\in \mathrm{C}, |h'|=1). \tag{3.35}$$

Notice that the function (3.35) is invariant under $a \to a^{-1}$. Hence we may assume $1 > a > 0$. Let

$$u = \left\{ \begin{array}{l} h \text{ as in (3.34)}\\ a \text{ as in (3.35)}\end{array}\right.$$

Thus $|u| < 1$, $u \in \mathbf{C}, u \neq 0$. Also let $z = h'$. Then both (3.34) and (3.35) may be expressed as

$$\begin{aligned}\frac{u}{(1-uz)(1-uz^{-1})} &= u\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}(uz)^m(uz^{-1})^n\\ &= \sum_{m,n=0}^{\infty} u^{m+n+1}z^{m-n}.\end{aligned}$$

Let $0 \leq k \in \mathbf{Z}$. We see that

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi}\frac{u}{(1-uz)(1-uz^{-1})}\overline{z}^k dt &= \sum_{m,n=0,m-n=k}^{\infty} u^{m+n+1} \qquad (z=e^{it})\\ &= \sum_{n=0}^{\infty} u^{2n+k+1} = -\frac{u^k}{u-u^{-1}}.\end{aligned}$$

Let Π' be the representation of G' which sends z to z^n, and let Π be the corresponding representation of G. We just computed that

$$\Theta_{\Pi}(h) = -\frac{h^k}{h-h^{-1}}, \tag{3.36}$$

$$\Theta_{\Pi}(g(a)) = -\frac{a^k}{a-a^{-1}}. \tag{3.37}$$

3.2 A Heuristic Computation

Suppose $G, G' \subset Sp$ is a dual pair with G' compact. Let Π, Π' be the representation of $\widetilde{G'}, \widetilde{G}$ as in Theorem 3.1.3. Recall the character formula

$$\Theta_\Pi(g) = \int_{\widetilde{G'}} \Theta(gg')\overline{\Theta_{\Pi'}(g')}dg' \qquad (g \in \widetilde{G}). \tag{3.38}$$

Let $(-1)\tilde{} \in \widetilde{Sp}$ be an element in the preimage of $-1 \in Sp$. We know from ???? that for $g \in \widetilde{Sp}^c$

$$\Theta_\Pi((-1)\tilde{}g) = T((-1)\tilde{}g)(0) = \Theta((-1)\tilde{})\int_W T(g)(w)dw. \tag{3.39}$$

Let $C_{\Pi'} \in \mathbf{C}^*$ be such that

$$\Pi'((-1)\tilde{}) = C_{\Pi'}I. \tag{3.40}$$

Then, by the left invariance of the Haar measure on $\widetilde{G'}$, (3.38) coincides with

$$\begin{aligned} \int_{\widetilde{G'}} \Theta((-1)\tilde{}gg')\overline{\Theta_{\Pi'}((-1)\tilde{}g')}dg' &= C_{\Pi'}\int_{\widetilde{G'}} \Theta((-1)\tilde{}gg')\overline{\Theta_{\Pi'}(g')}dg' && (3.41)\\ &= C_{\Pi'}\Theta((-1)\tilde{})\int_{\widetilde{G'}}\int_{G'\backslash W} T(gg')(\omega)\overline{\Theta_{\Pi'}(g')}d\dot{\omega}dg' && (3.42)\end{aligned}$$

Suppose, from now on, that G' is not compact. Recall the Weyl integration formula

$$\int_{\widetilde{G'}} f(g')dg' = \sum_{\widetilde{H'}} \frac{1}{|W(H')|}\int_{\widetilde{H'}} |\Delta(h')|^2 \int_{\widetilde{G'}\backslash\widetilde{H'}} f(g'h')d\dot{g}'dh'. \tag{3.43}$$

Let $\Theta_{\Pi'}$ be the character of an irreducible admissible representation Π' of $\widetilde{G'}$. For a Cartan subgroup H', let A' be the vector part and let T' be the compact part. Then $A \cong (\mathbf{R}^+)^p$, $T' \cong U_1^q$, for some p, q. Define

$$\Theta'_{\Pi'}(g) := C_{\Pi'}\Theta((-1)\tilde{})\sum_{\widetilde{H'}} \frac{1}{|W(H')|}\int_{\widetilde{H'}^{reg}} \overline{\Theta_{\Pi'}(h')}|\Delta(h')|^2 \int_{A'\backslash W} T(gh')d\dot{\omega}dh \quad (g \in \widetilde{G'}). \tag{3.44}$$

Then the Weyl intagration formula implies that

$$\Theta'_{\Pi'}(g) = C_{\Pi'}\Theta((-1)\tilde{})\int G'\overline{\Theta_{\Pi'}(g')}\int_{G'\backslash W} T(gg')(\omega)d\dot{\omega}dg'$$

Thus, if Π' coresoponds to Π via Howe's correspondence, we should have

$$\Theta'_{\Pi'} = \Theta_\Pi. \tag{3.45}$$

Chapter 4

The Cauchy Harish-Chandra Integral

4.1 The Wave Front Set

Fact 4.1.1 *Let* u *be a compactly supported distribution on* $\mathbf{R}^n$ *and let*

$$\mathcal{F}(u)(\xi) := \hat{u}(\xi) = \int_{\mathbf{R}^n} e^{-i2\pi\xi x} u(x)dx \qquad (\xi \in \mathbf{R}^n).$$

denote the Fourier transform of u*. Then* u *is a smooth function (times the Lebesgue measure) if and only if*

$$|\hat{u}(\xi)| \leq C_N (1 + |\xi|^2)^{-N} \qquad (\xi \in \mathbf{R}^n, N = 0, 1, 2, ...). \tag{4.1}$$

Proof:
We show the easy implication only. Suppose u is smooth, then with $\Delta = \sum_{j=1}^n \partial_{x_j}^2$,

$$(1 + |\xi|^2)^N \hat{u}(\xi) = ((1 - \Delta)^N u)\hat{}(\xi)$$

Thus,

$$(1 + |\xi|^2)^N |\hat{u}(\xi)| = |\int_{\mathbf{R}^n} e^{-i\xi(x)}((1 - \Delta)^N u)u(x)dx| \leq \int_{\mathbf{R}^n} |((1 - \Delta)^N u)u(x)|dx. \diamond$$

For a compactly supported distribution u in $\mathbf{R}^n$ define

$$\Sigma(u) \subset \mathbf{R}^n \setminus 0 \tag{4.2}$$

to be the complement of the union of open cones $V \subset \mathbf{R}^n \setminus 0$ such that

$$|\hat{u}(\xi)| \leq C_{N,V} (1 + |\xi|^2)^{-N} \qquad (\xi \in V,\ N = 0, 1, 2, ...). \tag{4.3}$$

Example 4.1.1 *Let* δ *be the Dirac delta at* $0 \in \mathbf{R}^n$*. Then*

$$\delta(\xi) = 1$$

thus $\Sigma(\delta) = \emptyset$.

Fact 4.1.2 *If* u *is a compctly supported distribution and* φ *is a compactly supported function on* $\mathbf{R}^n$*, then*

$$\Sigma(\varphi u) \subset \Sigma(u).$$

For u as above, (4.1.2), and for $x \in \mathbf{R}^n$ define

$$\Sigma_x(u) = \bigcap_{\varphi \in C_c^\infty(\mathbf{R}^n), \varphi(x) \neq 0} \Sigma(\varphi u). \tag{4.4}$$

Definition 4.1.1 *For any distribution u on an open subset of $\mathbf{R}^n$,*

$$WF(u) = \{(x, \zeta) \in \mathbf{R}^n \times (\mathbf{R}^n \setminus 0); \xi \in \Sigma_x(u)\}.$$

Example 4.1.2 *Let $n = 1$. Then*

$$WF(\delta) = \{0\} \times \mathbf{R}^n.$$

Let u be the inverse Fourier transfrom of the characteristic function of the interval $(0, \infty)$. Then

$$WF(u) = \{0\} \times \mathbf{R}^+.$$

For an open set $X \subset \mathbf{R}^n$ lat $\mathcal{D}'(X)$ denote the space of the distribution supported in X.

Definition 4.1.2 *Let $\Gamma \subset \mathbf{R}^n \times (\mathbf{R}^n \setminus 0)$ be a closed subset such that $(x, \xi) \in \Gamma \Leftrightarrow (x, t\xi)$ for all $t > 0$. Set*

$$\mathcal{D}'_\Gamma(X) = \{u \in \mathcal{D}'(X); WF(u) \subset \Gamma\}.$$

Let $u_j, u \in \mathcal{D}'_\Gamma(X)$. Then $u_j \to u$ in $\mathcal{D}'_\Gamma(X)$ if and only if

$$\lim_{j \to \infty} u_j(\varphi) = u(\varphi) \qquad (\varphi \in C_c^\infty(X)),$$

and

$$sup_j sup_{\xi \in V} |\xi|^N |(\varphi u)\hat{}(\xi)| < \infty$$

for all $N = 0, 1, 2, ...$ and all closed cones $V \subset \mathbf{R}^n \setminus 0$ and $\varphi \in C_c^\infty(X)$ such that

$$\Gamma \cap (supp\varphi \times V) = \emptyset.$$

Fact 4.1.3 *$C^\infty(X)$ is dense in $\mathcal{D}'_\Gamma(X)$.*

Let $X \subset \mathbf{R}^m, y \subset \mathbf{R}^n$ be open sets and let $f : X \to Y$ be a smooth map. Then the derivation

$$f'(x) = \{\frac{\partial f_i}{\partial x_i}\}_{i=1,2,...j=1,2,...}$$

The conormal bundle to f is

$$\mathcal{N}_f := \{(f(x), y) \in Y \times (\mathbf{R}^n \setminus 0); f'(x)^T(\eta) = 0\}.$$

For a subset $\Gamma \subset X \times (\mathbf{R}^n \setminus 0)$ let

$$f^*\Gamma = \{(x, f'(x)^T(\eta)); (f(x), \eta) \in \Gamma\}.$$

Theorem 4.1.1 *Let $\Gamma \subset Y \times (\mathbf{R}^m \setminus 0)$ be a non-empty, closed conic set such that*

$$\mathcal{N}_f \cap \Gamma = \emptyset.$$

Then the map

$$C^\infty(Y) \ni u \mapsto u \circ f \in C^\infty(X)$$

extends uniquelly to a continuosly map

$$\mathcal{D}'_\Gamma(Y) \ni u \mapsto f^*u \in \mathcal{D}'_{f^*\Gamma}(X).$$

Corollary 4.1.1 *The notion of the wave front set makes sense for distributions on real manifolds.*

4.2 The Cauchy-Harish Chandra Integral: $\int_{A'\backslash W} T(gh')(w)d\dot{w}$.

Fact 4.2.1 *Let $H' = A'T'$ be a Cartan subgroup of G'. Let A'' be the centralizer of A' in Sp and let A''' be the centralizer of A'' in Sp. If $A' \cong (\mathbf{R}^+)^p$, then $A''' \cong (\mathbf{R}^+)^p$. Moreover, (A''', A'') is a dual pair in Sp.*

Example 4.2.1 *$G' = Sp_4(\mathbf{R}), G = O_2$;*

$$\begin{aligned}
A' &= \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a > 0\}. \\
A''' &= \{\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}; a \in \mathbf{R}\backslash 0\} \in Sp_4(\mathbf{R}) \\
A'' &\cong Gl_2(\mathbf{R}) \\
W \ni \begin{pmatrix} w_1 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ w_2 \end{pmatrix} &\to \begin{pmatrix} w_1 g \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ w_2 (g^T)^{-1} \end{pmatrix} \in W, g \in GL_2(\mathbf{R}) \\
(A''', A'') &\cong (GL_1(\mathbf{R}), GL_2(\mathbf{R})).
\end{aligned}$$

Fact 4.2.2 *For any $E \in C_c^\infty(\overline{A''})$ the formula*

$$\int_{A'''\backslash W_{A'''}} \int_{\widetilde{A''}} E(g)T(g)(w)dgd\dot{w}$$

defines a distribution on $\widetilde{A''}$. (Here every consecutive integral is absolutely convergent.)

Fact 4.2.3 *Fix $h' \in \widetilde{H'}^{reg}$. The intersection of the WF set of the distribution (4.2.2) with the conormal bundle to the embedding*

$$\widetilde{G} \ni g \to h'g \in \widetilde{A''}$$

is empty. Thus, by (???),

$$\int_{A''\backslash W_{A'''}} T(h'g)(w)\dot{w} \qquad (g \in \widetilde{G})$$

exists as a distribution on $\widetilde{G}$.

Example 4.2.2 *$G' = Sp_2(\mathbf{R}), G = O_2$.*

$$\int_{A'''\backslash W_{A'''}} T(g)(w)d\dot{w} = det^{1/2}(g)\delta(\det(g+1)_W) \qquad G \in A'' = GL_4(\mathbf{R}).$$

If $h' = h'(a) = \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix} \in G'$ and $g \in O_2$ then

$$\det(h'(a)g + 1)_W = 0, \text{ if } a \neq 1.$$

Thus

$$\int_{A'''\backslash W_{A'''}} T(h'(a)g)(w)d\dot{w} = 0 \qquad (a \neq 1, g \in G).$$

Let h' be a regular element of the compact Cartan subgroup of G', as in (???). Let $h \in SO_2 \subset O_2$. Thus

$$\Theta(\widetilde{(-1)}) \int_W T(h'h)(w)dw = \Theta(h'h) = \frac{uz}{1-u^2z^2}\frac{uz^{-1}}{1-u^2z^{-2}},$$

where $h' \cong u^2,\ h \cong z^2 : u, z \in \mathbf{C}^*; |u| = 1$. Moreover,

$$\begin{aligned}
\Delta(h')\Theta(hh') &= (u^2 - u^{-2})\frac{uz}{1-u^2z^2}\frac{uz^{-1}}{1-u^2z^{-2}} \\
&= \frac{u^4-1}{(u^2-z^2)(u^2-z^{-2})} \\
&= \left(1 - \frac{1}{1-u^2z^2} - \frac{1}{1-u^2z^{-2}}\right) \\
&= 1 - \sum_{n=0}^{\infty} u^{2n}z^{2n} - \sum_{n=0}^{\infty} u^{2n}z^{-2n}.
\end{aligned}$$

Hence, for $n > 0$

$$\int_{|u|=1} -u^{-2m}\Delta(h')\Theta(hh') = \int_{|u|=1} -u^{-2m}(1 - \sum_{n=0}^{\infty} u^{2n}z^{2n} - \sum_{n=0}^{\infty} u^{2n}z^{-2n})$$

$$= z^{-2m} + z^{2m} = \Theta_{\Pi}(h).$$

Thus the Cauchy Harish-Chandra integral maps $\Theta_{\Pi'}$ to Θ_{Π}, and then provides symmetry, consistent with the First fundamental Theorem of Classical Invariant Theory, on the level of characters.

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