

Homework 8, solutions

1. (exercise 3 (a), page 341 in the book) Assume $x_n, y_n \in H$ are in the closed unit ball and $(x_n, y_n) \rightarrow 1$ if $n \rightarrow \infty$. Prove that

$$\|x_n - y_n\| \rightarrow 0, \text{ if } n \rightarrow \infty.$$

Since,

$$|(x_n, y_n)| \leq \|x_n\| \|y_n\| \leq 1,$$

we see that $\|x_n\| \rightarrow 1$ and $\|y_n\| \rightarrow 1$, if $n \rightarrow \infty$. Hence,

$$\|x_n - y_n\|^2 = \|x_n\|^2 + \|y_n\|^2 - 2\operatorname{Re}(x_n, y_n) \rightarrow 1 + 1 - 2 = 0$$

2. (exercise 3 (b), page 341 in the book) Assume $x_n \in H$, $x_n \rightarrow x$ weakly and $\|x_n\| \rightarrow \|x\|$, if $n \rightarrow \infty$. Prove that

$$\|x_n - x\| \rightarrow 0, \text{ if } n \rightarrow \infty.$$

Since, $x_n \rightarrow x$ weakly, $(x_n, x) \rightarrow (x, x) = \|x\|^2$. Thus

$$\|x_n - x\|^2 = \|x_n\|^2 + \|x\|^2 - 2\operatorname{Re}(x_n, x) \rightarrow \|x\|^2 + \|x\|^2 - 2\operatorname{Re} \|x\|^2 = 0.$$

3. Give an example of a Hilbert space H and an operator $T \in \mathcal{B}(H)$ such that T is not hermitian but T is normal.

Any 2 by 2, unitary matrix T , which is not hermitian will do, because $T^* = T^{-1}$ and $TT^{-1} = T^{-1}T = I$. For example

$$T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

4. Give an example of a Hilbert space H and an operator $T \in \mathcal{B}(H)$ such that T is not normal.

Let $H = \mathbb{C}^2$ with the usual scalar product and let T represent multiplication by the matrix

$$T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ & & 1 \end{pmatrix}.$$

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Then

$$T^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

and

$$TT^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

but

$$T^*T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$