Functional Analysis II, Spring 2008

Homework 8, solutions

1. (exercise 3 (a), page 341 in the book) Assume \(x_n, y_n \in H\) are in the closed unit ball and \((x_n, y_n) \to 1\) if \(n \to \infty\). Prove that

\[
\| x_n - y_n \| \to 0, \quad \text{if} \quad n \to \infty.
\]

Since,

\[
|\langle x_n, y_n \rangle| \leq \| x_n \| \| y_n \| \leq 1,
\]

we see that \(\| x_n \| \to 1\) and \(\| y_n \| \to 1\), if \(n \to \infty\). Hence,

\[
\| x_n - y_n \|^2 = \| x_n \|^2 + \| y_n \|^2 - 2\text{Re} \langle x_n, y_n \rangle \to 1 + 1 - 2 = 0
\]

2. (exercise 3 (b), page 341 in the book) Assume \(x_n \in H\), \(x_n \to x\) weakly and \(\| x_n \| \to \| x \|\), if \(n \to \infty\). Prove that

\[
\| x_n - x \| \to 0, \quad \text{if} \quad n \to \infty.
\]

Since, \(x_n \to x\) weakly, \(\langle x_n, x \rangle \to \langle x, x \rangle = \| x \|^2\). Thus

\[
\| x_n - x \|^2 = \| x_n \|^2 + \| x \|^2 - 2\text{Re} \langle x_n, x \rangle \to \| x \|^2 + \| x \|^2 - 2\text{Re} \| x \|^2 = 0.
\]

3. Give an example of a Hilbert space \(H\) and an operator \(T \in B(H)\) such that \(T\) is not hermitian but \(T\) is normal.

Any 2 by 2, unitary matrix \(T\), which is not hermitian will do, because \(T^* = T^{-1}\) and \(TT^{-1} = T^{-1}T = I\). For example

\[
T = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.
\]

4. Give an example of a Hilbert space \(H\) and an operator \(T \in B(H)\) such that \(T\) is not normal.

Let \(H = \mathbb{C}^2\) with the usual scalar product and let \(T\) represent multiplication by the matrix

\[
T = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]
Then

\[ T^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \]

and

\[ TT^* = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \]

but

\[ T^*T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}. \]