Functional Analysis II, Spring 2008

Homework 7, solutions

1. (exercise 5, page 301 in the book) Suppose A is a commutative Banach algebra, $x \in A$ and f is a holomorphic function in some open set $\Omega \subseteq \mathbb{C}$ that contains the range of $\hat{x}$. Prove that there exists $y \in A$ such that $\hat{y} = f \circ \hat{x}$. Prove that y is uniquely determined by x and f if A is semisimple.

In terms of section 10.26 in the book, let

$$y = \tilde{f}(x) = \frac{1}{2\pi i} \int_\gamma f(\lambda)(\lambda e - x)^{-1} \, d\lambda.$$

Then $y \in A$ and for $h \in \Delta$,

$$h(y) = \frac{1}{2\pi i} \int_\gamma f(\lambda) h((\lambda e - x)^{-1}) \, d\lambda = \frac{1}{2\pi i} \int_\gamma f(\lambda)(\lambda - h(x))^{-1} \, d\lambda = f(h(x)).$$

If A is semisimple, $y, z \in A$ and $h(y) = h(z)$ for all $h \in \Delta$, then $y = z$. Thus, in this case, the y is uniquely determined by f and x.

2. Find an example of $x \in L^1(\mathbb{Z})$ such that $\| x * x \| < \| x \|^2$.

Let $x = \delta_1 + \delta_0 - \delta_{-1}$. Then

$$x * x = \delta_{-2} - 2\delta_{-1} - \delta_0 + 2\delta_1 + \delta_2.$$

Hence,

$$\| x * x \| = 1 + 2 + 1 + 2 + 1 = 7 < 9 = (1 + 1 + 1)^2 = \| x \|^2 .$$