Functional Analysis II, Spring 2008

Homework 6, solutions

1. Give an example of a Banach algebra whose radical has dimension 6.

Let A be the subalgebra of $M_{4,4}(\mathbb{C})$ consisting of all the matrices of the form

$$\begin{pmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & g \\ 0 & 0 & 0 & a \end{pmatrix} \qquad (a, b, c, d, e, f, g \in \mathbb{C}).$$

Then $\Delta(A)$ consists of one element h only,

$$h : \begin{pmatrix} a & b & c & d \\ 0 & a & e & f \\ 0 & 0 & a & g \\ 0 & 0 & 0 & a \end{pmatrix} \to a \in \mathbb{C}.$$

Hence the radical of A is equal to the kernel of h, which has dimension 6.

Compute the spectrum $\Delta(A)$ of the following Banach algebras

2. $A = L^1(\mathbb{Z}^N)$, $N = 1, 2, 3, ...$

Let $\mathbb{T} = \{z \in \mathbb{C};\ |z| = 1\}$. For $z = (z_1, z_2, ..., z_N) \in \mathbb{T}^N$ and $x \in A$ let

$$\mathcal{F}x(z) = \sum_{n \in \mathbb{Z}^N} x(n) z^n$$

where $n = (n_1, n_2, ..., n_N)$ and $z^n = z_1^{n_1} z_2^{n_2} ... z_N^{n_N}$. Let

$$h_z(x) = \mathcal{F}x(z) \qquad (z \in \mathbb{T}^N,\ x \in A).$$

The argument used in class for $N = 1$ can be generalized to show that

$$\Delta(A) = \{h_z;\ z \in \mathbb{T}^N\}.$$

3. $A = L^1(\mathbb{R}^N) + \mathbb{C}\delta$, $N = 1, 2, 3, ...$

Recall the Fourier transform

$$\mathcal{F}x(t) = \int_{\mathbb{R}^N} x(s)\, exp(-it \cdot s)\, ds \qquad (x \in L^1(\mathbb{R}^N), t \in \mathbb{R}^N),$$

where $t \cdot s = t_1 s_1 + t_2 s_2 + ... + t_N s_N$. Define

$$h_t(x + \alpha\delta) = \mathcal{F}x(t) + \alpha \qquad (x \in L^1(\mathbb{R}^N), \alpha \in \mathbb{C}, t \in \mathbb{R}^N),$$

and

$$h_\infty(x + \alpha\delta) = \alpha \qquad (x \in L^1(\mathbb{R}^N), \alpha \in \mathbb{C}).$$

The argument used in class for $N = 1$ can be generalized to show that

$$\Delta(A) = \{h_t;\ t \in \mathbb{R}^N\} \cup \{h_\infty\}.$$