Functional Analysis II, Spring 2008

Homework 4, solutions

1. In the Banach algebra $L^1(\mathbb{Z}/N\mathbb{Z})$ find $\sqrt{\delta_1}$. (Here, $\delta_1(n) = 1$ if $n = 1$ and zero otherwise.)

Let $\omega = e^{\frac{2\pi i}{N}}$. Recall the Fourier transform

$$\mathcal{F}u(n) = \sum_{m=0}^{N-1} u(m)\,\omega^{-mn},$$

and the inverse Fourier transform

$$\mathcal{F}^{-1}v(m) = \frac{1}{N}\sum_{m=0}^{N-1} v(n)\,\omega^{mn}.$$

Also, $\mathcal{F}$ transforms the convolution into the product of functions. Thus

$$\sqrt{\delta_1} = \mathcal{F}^{-1}\left(\sqrt{\mathcal{F}\delta_1}\right). \tag{1.1}$$

(Notice, by the way that δ_1 is NOT the identity in $L^1(\mathbb{Z}/N\mathbb{Z})$. The identity is δ_0.) Since,

$$\mathcal{F}\delta_1(m) = \sum_{k=0}^{N-1} \delta_1(k)\omega^{-km} = \omega^{-m},$$

we see that

$$\sqrt{\mathcal{F}\delta_1}(m) = \omega^{-\frac{m}{2}}.$$

Therefore (1.1) implies

$$\sqrt{\delta_1}(n) = \frac{1}{N}\sum_{m=0}^{N-1} \omega^{-\frac{m}{2}}\omega^{mn} = \frac{1}{N}\sum_{m=0}^{N-1}\left(\omega^{(n-\frac{1}{2})}\right)^m = \frac{1}{N}\,\frac{\omega^{(nN-\frac{N}{2})}-1}{\omega^{(n-\frac{1}{2})}-1}.$$

2. Define the natural logarithm as

$$ln(z) = \int_1^z \frac{1}{\lambda}\,d\lambda \qquad (z \in \mathbb{C}\setminus(-\infty, 1]).$$

Compute the limits

$$\lim_{y\to 0+} ln(x+iy), \quad \lim_{y\to 0-} ln(x+iy) \qquad (x\in\mathbb{R}\setminus\{0\})$$

Answer:

$$\lim_{y\to 0+} ln(x+iy) = \begin{cases} ln(|x|) \text{ if } x>0, \\ ln(|x|)+i\pi \text{ if } x<0, \end{cases}$$

$$\lim_{y\to 0+} ln(x+iy) = \begin{cases} ln(|x|) \text{ if } x>0, \\ ln(|x|)-i\pi \text{ if } x<0, \end{cases}$$

3. Find all the complex numbers z such that $e^z = 1$.

Answer: $z = k\,2\pi i$, $k \in \mathbb{Z}$.