Functional Analysis II, Spring 2008

Homework 2, solutions

Recall the finite cyclic group $\mathbb{Z}/N\mathbb{Z} = \{0, 1, 2, ..., N-1\}$ with the addition modulo N. For $x, y \in L^1(\mathbb{Z}/N\mathbb{Z})$ define the convolution

$$x * y(n) = \sum_{m=0}^{N-1} x(m)y(n-m) \qquad (n = 0, 1, 2, ...N-1)$$

and the Fourier transform

$$\hat{x}(n) = \sum_{m=0}^{N-1} x(m)e^{-\frac{2\pi i}{N}mn} \qquad (n = 0, 1, 2, ...N-1).$$

1. Show that

$$\| x * y \|_1 \leq \| x \|_1 \| y \|_1 \qquad (x, y \in L^1(\mathbb{Z}/N\mathbb{Z})),$$

and

$$x * \delta_0 = x \qquad (x \in L^1(\mathbb{Z}/N\mathbb{Z})).$$

(This with a few more steps shows that $L^1(\mathbb{Z}/N\mathbb{Z})$, with the convolution and the L^1 norm is a Banach algebra.)

Since,

$$\| x * y \|_1 = \sum_n |x * y(n)| = \sum_n |\sum_m x(m)y(n-m)|$$
$$\leq \sum_{m,n} |x(m)||y(n-m)| = \sum_{m,n} |x(m)||y(m)| \leq \| x \|_1 \| y \|_1,$$

the first part follows. The second statement is obvious.

2. Consider the Banach algebra $A = L^1(\mathbb{Z}/N\mathbb{Z})$. Show that $x \in A$ is invertible if and only if $\hat{x}(n) \neq 0$ for all n.

Let B denote the Banach algebra of all functions $f : \mathbb{Z}/N\mathbb{Z} \to \mathbb{C}$, with the maximum norm. Then the Fourier transform

$$\mathcal{F} : A \ni x \to \hat{x} \in B$$

is a linear bijection which maps the convolution $x * y$ to the product $\hat{x}\hat{y}$. Therefore $x \in A$ is invertible if and only if $\hat{x} \in B$ is invertible. Since the identity in B is the constant function equal 1, the later happens if and only if $\hat{x}$ has no zeros.

3. Show that for $x \in A$ (as above) the spectrum of x is equal to the set of all the values of the Fourier transform $\hat{x}$: $\sigma(x) = \{\hat{x}(n);\ n = 0, 1, 2, ..., N-1\}$.

The know from previous problem that $x \in A$ is invertible if and only if $\hat{x} \in B$ is invertible. Hence, $\lambda\delta_0 - x \in A$ is invertible if and only if $\lambda 1 - \hat{x} \in B$ is invertible. The later happens if and only if the function $\lambda 1 - \hat{x}$ has no zeros. This is equivalent to λ not being in the range of $\hat{x}$.

4. Think of Problem 19, page 273 in Rudin's book

We solved it in class.