Functional Analysis II, Spring 2008

Homework 1, solutions

1. Let A be a Banach algebra and let $x, y \in A$ be two invertible elements. Show that the product xy is also invertible.

Since x and y are invertible $y^{-1}x^{-1} \in A$, $xyy^{-1}x^{-1} = xx^{-1} = e$ and $y^{-1}x^{-1}xy = yy^{-1} = e$. Thus $y^{-1}x^{-1}$ is the inverse of xy.

2. Give an example of a Banach algebra A and two elemens $x, y \in A$ such that $xy = e$ but $yx \neq e$.

Recall the set of natural numbers $\mathbb{N} = \{1, 2, 3, ...\}$. Let $A = B(L^2(\mathbb{N}))$, where $L^2(\mathbb{N})$ is the space of square integrable functions $v : \mathbb{N} \to \mathbb{C}$. Define $x, y \in A$ by

$$yv(1) = 0,\ yv(2) = v(1),\ yv(3) = v(2),\ yv(4) = v(3), ... ,$$
$$xv(1) = v(2),\ xv(2) = v(3),\ xv(3) = v(4),\ xv(4) = v(5), ...$$

Then

$$x(yv)(n) = yv(n+1) = v(n) \qquad (n \in \mathbb{N}).$$

However,

$$y(xv)(1) = 0,$$

for every v. Thus $xyv = v$ for all v, but if $v(1) \neq 0$ then $yxv \neq v$. Therefore, $xy = e$ but $yx \neq e$

3. Show that in a finite dimensional Banach algebra A for any two elemens $x, y \in A$ if $xy = e$ then $yx = e$.

Recall that A is isomorphic with a Banach subalgebra $\tilde{A} \subseteq B(A)$. Since the space A has dimension $n < \infty$, the algebra $B(A)$ is isomorphic to the algebra $M_n(\mathbb{C})$ of n by n matrices with complex entries. We know from linear algebra that a matrix a is invertible if and only if the determinand $det(a) \neq 0$. However

$$det(ab) = det(a)det(b) = det(b)det(a) = det(ba) \qquad (a, b \in M_n(\mathbb{C})).$$

Hence for any two elements $\tilde{x}, \tilde{y} \in \tilde{A}$, $\tilde{x}\tilde{y}$ is invertible if and only if $\tilde{y}\tilde{x}$ is invertible. Thus, by the above mentioned isomorphism, xy is invertible if and only if yx is invertible.

4. Consider the Banach algebra $A = L^1(\mathbb{Z})$. Let $x \in A$ be defined by $x(0) = 1$, $x(1) = \frac{1}{2}$ and $x(n) = 0$ for all $n \notin \{0, 1\}$. Find the inverse x^{-1} in A.

Let

$$\delta_k(n) = \begin{cases} 1 \text{ if } n = k, \\ 0 \text{ if } n = k. \end{cases}$$

Then each $\delta_k \in A$, $\delta_0 = e$ is the identity and

$$x = \delta_0 + 2^{-1}\delta_1 = \delta_0 - (-2^{-1})\delta_1..$$

notice that

$$\delta_k * \delta_l = \delta_{k+l} \qquad (k, l \in \mathbb{Z}).$$

Furthermore, $x = e - (-2^{-1})\delta_1$ and $\| (-2^{-1})\delta_1 \| = 2^{-1} < 1$ Hence, the inverse x^{-1} exists and is equal to

$$\begin{aligned} x^{-1} =& (e - (-2^{-1})\delta_1)^{-1} = \delta_0 + (-1)2^{-1}\delta_1 + (-1)^2 2^{-2}\delta_2 + (-1)^3 2^{-3}\delta_3 + ... \\ =& \delta_0 - 2^{-1}\delta_1 + 2^{-2}\delta_2 - 2^{-3}\delta_3 + ... \end{aligned}$$

In other words,

$$x^{-1}(n) = \begin{cases} (-1)^n 2^{-n} \text{ if } n \geq 0, \\ 0 \text{ if } n < 0. \end{cases}$$

5. Give an example of a Banach algebra A and an element $x \in A$ such that $x \neq 0$ but $x^2 = 0$.

Let $A = M_2(\mathbb{C})$ and

$$x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Then A, with the operator norm, is a Banach algebra and $x \in A$ has the required properties.