1. Let $A$ be a Banach algebra and let $x, y \in A$ be two invertible elements. Show that the product $xy$ is also invertible.

Since $x$ and $y$ are invertible, $y^{-1}x^{-1} \in A$, $xyy^{-1}x^{-1} = xx^{-1} = e$ and $y^{-1}x^{-1}xy = yy^{-1} = e$. Thus $y^{-1}x^{-1}$ is the inverse of $xy$.

2. Give an example of a Banach algebra $A$ and two elements $x, y \in A$ such that $xy = e$ but $yx \neq e$.

Recall the set of natural numbers $\mathbb{N} = \{1, 2, 3, \ldots\}$. Let $A = B(L^2(\mathbb{N}))$, where $L^2(\mathbb{N})$ is the space of square integrable functions $v : \mathbb{N} \to \mathbb{C}$. Define $x, y \in A$ by

\[ yv(1) = 0, \quad yv(2) = v(1), \quad yv(3) = v(2), \quad yv(4) = v(3), \ldots, \]
\[ xv(1) = v(2), \quad xv(2) = v(3), \quad xv(3) = v(4), \quad xv(4) = v(5), \ldots. \]

Then

\[ x(yv)(n) = yv(n + 1) = v(n) \quad (n \in \mathbb{N}). \]

However,

\[ y(xv)(1) = 0, \]

for every $v$. Thus $xyv = v$ for all $v$, but if $v(1) \neq 0$ then $yxv \neq v$. Therefore, $xy = e$ but $yx \neq e$.

3. Show that in a finite dimensional Banach algebra $A$ for any two elements $x, y \in A$ if $xy = e$ then $yx = e$.

Recall that $A$ is isomorphic with a Banach subalgebra $\tilde{A} \subseteq B(A)$. Since the space $A$ has dimension $n < \infty$, the algebra $B(A)$ is isomorphic to the algebra $M_n(\mathbb{C})$ of $n \times n$ matrices with complex entries. We know from linear algebra that a matrix $a$ is invertible if and only if the determinand $det(a) \neq 0$. However

\[ det(ab) = det(a)det(b) = det(b)det(a) = det(ba) \quad (a, b \in M_n(\mathbb{C})). \]

Hence for any two elements $\tilde{x}, \tilde{y} \in \tilde{A}$, $\tilde{x}\tilde{y}$ is invertible if and only if $\tilde{y}\tilde{x}$ is invertible. Thus, by the above mentioned isomorphism, $xy$ is invertible if and only if $yx$ is invertible.

4. Consider the Banach algebra $A = L^1(\mathbb{Z})$. Let $x \in A$ be defined by $x(0) = 1$, $x(1) = \frac{1}{2}$ and $x(n) = 0$ for all $n \notin \{0, 1\}$. Find the inverse $x^{-1}$ in $A$.

Let

\[ \delta_k(n) = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{if } n = k. \end{cases} \]

Then each $\delta_k \in A$, $\delta_0 = e$ is the identity and

\[ x = \delta_0 + 2^{-1}\delta_1 = \delta_0 - (-2^{-1})\delta_1. \]
notice that 

\[ \delta_k \ast \delta_l = \delta_{k+l} \quad (k, l \in \mathbb{Z}). \]

Furthermore, \( x = e - (-2^{-1})\delta_1 \) and \( \| (-2^{-1})\delta_1 \| = 2^{-1} < 1 \) Hence, the inverse \( x^{-1} \) exists and is equal to

\[
x^{-1} = (e - (-2^{-1})\delta_1)^{-1} = \delta_0 + (-1)2^{-1}\delta_1 + (-1)^22^{-2}\delta_2 + (-1)^32^{-3}\delta_3 + ... = \delta_0 - 2^{-1}\delta_1 + 2^{-2}\delta_2 - 2^{-3}\delta_3 + ... .
\]

In other words,

\[
x^{-1}(n) = \begin{cases} 
(-1)^n2^{-n} & \text{if } n \geq 0, \\
0 & \text{if } n < 0.
\end{cases}
\]

5. Give an example of a Banach algebra \( A \) and an element \( x \in A \) such that \( x \neq 0 \) but \( x^2 = 0 \).

Let \( A = M_2(\mathbb{C}) \) and

\[
x = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
\]

Then \( A \), with the operator norm, is a Banach algebra and \( x \in A \) has the required properties.