Functional Analysis II, Spring 2008

Final Exam, solutions

1. Let H be a Hilbert space and let $T \in B(H)$ be invertible. Show that T has a unique polar decomposition

$$T = UP,$$

where $P \in B(H)$ is a positive operator and $U \in B(H)$ is a unitary operator.

See the book, Theorem 12.35.

2. Find the polar decomposition of the matrix

$$T = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

Use matlab or mathematica.

In the remaining problems, let $H = L^2(\mathbb{Z})$. Recall the translations

$$\rho(n)f(m) = f(m-n) \qquad (m, n \in \mathbb{Z}, f \in H).$$

3. Show that for each $\phi \in L^1(\mathbb{Z})$,

$$\sum_{n\in\mathbb{Z}} \phi(n)\rho(n) \in B(H) \text{ and } \| \sum_{n\in\mathbb{Z}} \phi(n)\rho(n) \| \leq \| \phi \|_1 .$$

Moreover, for any compactly supported f,

$$\sum_{n\in\mathbb{Z}} \phi(n)\rho(n)f(m) = \sum_{n\in\mathbb{Z}} \phi(n)f(m-n).$$

Thus

$$\sum_{n\in\mathbb{Z}} \phi(n)\rho(n) = conv(\phi).$$

We know from the last homework that each $\rho(n)$ is unitary. Hence $\| \rho(n) \| = 1$. Thus

$$\| \sum_{n\in\mathbb{Z}} \phi(n)\rho(n) \| \leq \sum_{n\in\mathbb{Z}} |\phi(n)| \, \| \rho(n) \| \leq \| \phi \|_1 .$$

The rest is obvious.

4. Let $A \subseteq B(H)$ be the smallest C^*-algebra generated by $conv(L^1(\mathbb{Z}))$. Find the spectrum of A and the corresponding resolution of the identity.

We know from the last homework that the spectrum of A coincides with $\mathbb{T}$. Furthermore, the resolution of identity is given by

$$E(\omega) = conv(\mathcal{F}^{-1}(\mathbb{I}_\omega)) \qquad (\omega \subseteq \mathbb{T}, \text{ measurable}).$$