Functional Analysis II, Spring 2008

Exam 1, solutions

1. Define a Banach algebra.

Look it up in the book.

2. Give an example of a commutative and a non-commutative Banach algebra.

The one dimensional algebra $A = \mathbb{C}$ is commutative and B, the Banach algebra of bounded operators on a Hilbert space of dimension larger than 1 is not.

3. Give an example of two Banach algebras A and B which are identical as algebras, but have two different norms.

Let A_1 be the Banach algebra of bounded operators on $\mathbb{C}^2$, equipped with the norm $\| z \|_1 = |z_1| + |z_2|$, and let A_2 be the Banach algebra of bounded operators on $\mathbb{C}^2$, equipped with the norm $\| z \|_1 = \sqrt{|z_1|^2 + |z_2|^2}$. Then as algebras both A_1 and A_2 are equal to the algebra of all linear maps from $\mathbb{C}^2$ to $\mathbb{C}^2$. Consider the following map

$$x : \mathbb{C}^2 \ni (z_1, z_2) \to (z_1 + z_2, z_1 - z_2) \in \mathbb{C}^2.$$

The norm of $x \in A_1$ is equal to

$$max\{\frac{|z_1 + z_2| + |z_1 - z_2|}{|z_1| + |z_2|};\ |z_1| + |z_2| = 1\} \geq \frac{1+1}{1} = 2,$$

but the norm of $x \in A_2$ is equal to

$$max\{\frac{\sqrt{|z_1 + z_2|^2 + |z_1 - z_2|^2}}{|z_1| + |z_2|};\ |z_1| + |z_2| = 1\} = \sqrt{2}.$$

4. Define the spectrum $\sigma(x)$ and the spectral radius $\rho(x)$ of an element x of a Banach algebra A.

Look it up in the book.

5. Give an example of a Banach algebra A and an element $x \in A$ such $\sigma(x) = \{z \in \mathbb{C};\ |z| = 1\}$.

Since the spectrum is infinite, the algebra has to be infinite dimensional. Consider for example $A = C(I)$, the algebra of continuous functions x from the unit interval $I = [0, 1]$ to $\mathbb{C}$. The spectrum of x is the same as the set of the values of x. Hence, for $x(t) = e^{2\pi it}$, we have $\sigma(x) = \{e^{2\pi it};\ 0 \leq t \leq 1\}$.

6. Suppose A is a finite dimensional Banach algebra. Show that for every $x \in A$ the spectrum $\sigma(x)$ is finite.

In this case, A is isomorphic to the Banach algebra of all operators $x : A \to A$ on the finite dimensional space A. Hence, the spectrum of any element $x \in A$ is equal to the set of the eigenvalues, which is finite.

7. Let $c(\lambda) = (\lambda+1)(\lambda-1)^{-1}$. In the Banach algebra $L^1(\mathbb{Z}/N\mathbb{Z})$ find $\tilde{c}(\frac{1}{2}\delta_1)$. (Here, $\delta_1(n) = 1$ if $n = 1$ and zero otherwise.)

Notice that for $|\lambda| < 1$,

$$c(\lambda) = -(\lambda+1)(1-\lambda)^{-1} = -(\lambda+1)\sum_{n=0}^{\infty}\lambda^n = -1 - \sum_{n=1}^{\infty} 2\lambda^n.$$

Also, $\| \frac{1}{2}\delta_1) \|_1 = \frac{1}{2}$. Hence,

$$\tilde{c}(\frac{1}{2}\delta_1) = -\delta_0 - \sum_{n=1}^{\infty} 2(\frac{1}{2}\delta_1)^{*n} = -\delta_0 - \sum_{n=1}^{\infty} 2^{1-n}\delta_n = \delta_0 - \sum_{n=0}^{\infty} 2^{1-n}\delta_n.$$

But, $\delta_n = \delta_{n+N}$. Therefore,

$$\sum_{n=0}^{\infty} 2^{1-n}\delta_n = \sum_{k=0}^{N-1}\left(\sum_{n=0}^{\infty} 2^{1-(k+nN)}\right)\delta_{k+nN}$$
$$= \sum_{k=0}^{N-1}\left(\sum_{n=0}^{\infty} 2^{1-(k+nN)}\right)\delta_k = \sum_{k=0}^{N-1} 2^{1-k}\frac{1}{1-2^{-N}}\delta_k.$$

Thus

$$\tilde{c}(\frac{1}{2}\delta_1) = \delta_0 - \sum_{k=0}^{N-1}\frac{2^{1-k}}{1-2^{-N}}\delta_k.$$

8. Is there a Banach algebra A and an element $x \in A$ such that $0 < \rho(x) = 2 \| x \|$?

No because $\rho(x) \leq \| x \|$.

9. Show that any two one-dimensional Banach algebras are isomorphic.

If $dim\, A = 1$, then the map

$$A \ni \lambda e \to \lambda \in \mathbb{C}$$

is an isomorphism of Banach algebras.

10. (Exercise 24 in the book) Prove that A is commutative if there is a constant $M < \infty$ such that $\| xy \| \leq M \| yx \|$ for all $x, y \in A$.

Let $f : A \to \mathbb{C}$ be a continuous linear functional and let $x, y \in A$. Then the function

$$\mathbb{C} \ni z \to f(e^{zx}ye^{-zx}) \in \mathbb{C}$$

is holomorphic and bounded, because

$$|f(e^{zx}ye^{-zx})| \leq \| f \| \| e^{zx}ye^{-zx} \| \leq \| f \| \; M \| e^{-zx}e^{zx}y \| = \| f \| \; M \| y \| .$$

Hence, by Liuoville's Theorem, it is constant. Thus the derivative is zero, which means that $f(xy) = f(yx)$. Since f is arbitrary, we see that $xy = yx$.

11. Prove that A is commutative if $\| x^2 \| = \| x \|^2$ for all $x \in A$.

The condition $\| x^2 \| = \| x \|^2$, together with the limit formula for the spectral radius, implies that

$$\rho(x) = \| x \| \qquad (x \in A).$$

Let $x, y \in A$. Then

$$(xy)^n = x(yx)^{n-1}y.$$

Hence, again by the limit formula, $\rho(xy) = \rho(yx)$. Therefore,

$$\| xy \| = \rho(xy) = \rho(yx) = \| yx \| .$$

Thus in previous problem we may take $M = 1$ and conclude that A is commutative.