Here is the missing part of the argument I gave today (11.27.2007).

Let $Y$ be a complete metric space and let $S \subseteq Y$ be a closed subset, which is totally bounded in the sense that for every $\epsilon > 0$ there are elements $s_1, s_2, \ldots, s_N$ in $S$ such that

$$S \subseteq \bigcup_{k=1}^{N} B_\epsilon(s_k).$$

Fix an infinite sequence $A = \{a_1, a_2, a_3, \ldots\}$ in $S$. For $n = 1, 2, 3, \ldots$ let us cover $S$ as above by choosing $\epsilon = \frac{1}{2^n}$,

$$S \subseteq \bigcup_{k=1}^{N_n} B_{\frac{1}{2^n}}(s_k^n).$$

Since $A$ is contained in $S$, there is $k$ such that the set

$$A_1 = A \cap B_{\frac{1}{2}}(s_k^1)$$

is infinite. Similarly, there is $k$ such that the set

$$A_2 = A_1 \cap B_{\frac{1}{2^2}}(s_k^2)$$

is infinite. We continue this way inductively and obtain a descending sequence of sets

$$A \supseteq A_1 \supseteq A_2 \supseteq \ldots \supseteq A_n \supseteq \ldots,$$

with the diameter of $A_n$ smaller than $\frac{1}{n}$. Pick $x_n \in A_n$. Then $x_1, x_2, \ldots$ is a subsequence of the sequence $a_1, a_2, \ldots$, which is also a Cauchy sequence. Since $Y$ is complete and $S$ is closed, there is $x \in S$ such that

$$x = \lim_{n \to \infty} x_n.$$

This shows that $S$ is compact.

This completes the argument that a limit of a sequence of compact operators, in norm topology, is a compact operator.

Problems ($X, Y, Z$ denote Banach spaces).

1. Prove that $T \in B(X,Y)$ be a surjective isometry if and only if $T^* \in B(Y^*, X^*)$ is a surjective isometry.

Suppose $T \in B(X,Y)$ be a surjective isometry. Then

$$\| T^* y^* \| = \sup \{|\langle x, T^* y^* \rangle|; \ x \in X, \| x \| \leq 1\}$$

$$= \sup \{|\langle Tx, y^* \rangle|; \ x \in X, \| x \| \leq 1\}$$

$$= \sup \{|\langle y, y^* \rangle|; \ y \in Y, \| y \| \leq 1\}$$

$$= \| y^* \|.$$
Hence, $T^*$ is an isometry. Let $x^* \in X^*$. Then
\[ y^* = (T^{-1})^* x^* \in Y^* \] and $T^* y^* = y^* \circ T = x^* \circ T^{-1} \circ T = x^*$.

Therefore, $T$ is surjective.

Suppose $T^* \in B(Y^*, X^*)$ be a surjective isometry. Then
\[ \| Tx \| = \sup \{|(Tx, y^*)|; \, y^* \in Y^*, \| y^* \| \leq 1\} \]
\[ = \sup \{|(x, T^* y^*)|; \, y^* \in Y^*, \| y^* \| \leq 1\} \]
\[ = \sup \{|(x, x^*)|; \, x^* \in X^*, \| x^* \| \leq 1\} \]
\[ = \| x \|. \]

Hence, $T^*$ is an isometry. By Corollary of Theorem 4.12, the range of $T$, $R(T)$ is dense in $Y$. Since $T$ is an isometry, $R(T)$ is a complete subspace of $Y$ (every Cauchy sequence of elements of $R(T)$ converges to an element of $R(T)$). Hence, $R(T)$ is a closed subspace of $Y$. Thus $R(T) = Y$. Therefore $T$ is surjective.

2. Let $\sigma$, $\tau$ be the weak* topologies of $X^*$ and $Y^*$, respectively. Prove that $S$ is a continuous linear mapping of $(Y^*, \tau)$ into $(X^*, \sigma)$ if and only if $S = T^*$ for some $T \in B(X, Y)$.

Let $S = T^*$ for some $T \in B(X, Y)$. A typical open neighborhood of zero in the weak* topology in $X^*$ looks as follows
\[ V = \{x^* \in X^*; |\langle x_j, x^* \rangle| < r_j \text{ for all } 1 \leq j \leq n\} \]
where $x_1, x_2, ..., x_n \in X$. Then,
\[ S^{-1}(V) = \{y^* \in Y^*; |\langle x_j, S y^* \rangle| < r_j \text{ for all } 1 \leq j \leq n\} \]
\[ = \{y^* \in Y^*; |\langle T x_j, y^* \rangle| < r_j \text{ for all } 1 \leq j \leq n\}, \]
which is an open neighborhood of zero in the weak* topology in $Y^*$. Thus $S$ is continuous.

Suppose, $S$ is a continuous linear mapping of $(Y^*, \tau)$ into $(X^*, \sigma)$. Then for every $x \in X$,
\[ Y^* \ni y^* \mapsto \langle x, S y^* \rangle \in \mathbb{C} \]
is a continuous linear map in the weak* topology. Hence, by Theorem 3.10, there is $y \in Y$ such that
\[ (2.1) \quad \langle x, S y^* \rangle = \langle y, y^* \rangle \quad (y^* \in Y^*). \]

The element $y$ in (2.1) is uniquely determined by $x$, and it is easy to check that the function $T : x \to y$ is linear. Furthermore,
\[ \sup \{|\langle x, S y^* \rangle|; x \in X, \| x \| \leq 1\} = \| S y^* \| < \infty \quad (y^* \in Y^*). \]

Hence, by Banach Steinhaus Theorem, 2.6,
\[ \sup \{|\langle x, S y^* \rangle|; x \in X, \| x \| \leq 1, y^* \in Y^*, \| y^* \| \leq 1\} < \infty. \]
In other words.

\[ \sup \{ \|Tx, y^*\| : x \in X, \|x\| \leq 1, y^* \in Y^*, \|y^*\| \leq 1 \} < \infty, \]

which means that the map \( T \) is bounded.

3. Prove that if \( T \in B(X, Y) \) and \( S \in B(Y, Z) \), then \((ST)^* = T^*S^*\).

Since for \( x^* \in X^* \), \( T^*(x^*) = x^* \circ T \), we see that \((ST)^*x^* = x^* \circ (ST) = x^* \circ (S \circ T) = (x^* \circ S) \circ T = T^*(S^*(x^*))\).

4. Suppose \( \mu \) is a finite positive measure on a measure space \( \Omega \), \( \mu \times \mu \) is the corresponding product measure on \( \Omega \times \Omega \) and \( K \in L^2(\mu \times \mu) \). Define

\[ (T_K f)(s) = \int_{\Omega} K(s, t)f(t) \, d\mu(t) \quad (f \in L^2(\mu)). \]

(a) Prove that \( T_K \in B(L^2(\mu)) \) and

\[ \|T_K\|^2 \leq \int_{\Omega} \int_{\Omega} |K(s, t)|^2 \, d\mu(s) \, d\mu(t). \]

(b) Suppose \( a_i, b_i \in L^2(\mu) \), for \( 1 \leq i \leq n \). Put \( K_n(s, t) = \sum_i a_i(s)b_i(t) \) and define \( T_{K_n} \) in terms of \( K_n \). Prove that \( \dim R(T_{K_n}) \leq n \).

(c) Deduce that \( T \) is a compact operator.

(e) Describe the adjoint of \( T \).

Since, by Cauchy-Schwartz inequality,

\[ \|T_K f\|^2 = \int_{\Omega} |Tf(s)|^2 \, d\mu(s) \]

\[ = \int_{\Omega} \left( \int_{\Omega} |K(s, t)f(t) \, d\mu(t)| \right)^2 \, d\mu(s) \]

\[ \leq \int_{\Omega} \left( \int_{\Omega} |K(s, t)|^2 \, d\mu(t) \right)^{1/2} \left( \int_{\Omega} |f(t)|^2 \, d\mu(t) \right)^{1/2} \, d\mu(s) \]

\[ = \int_{\Omega} \left( \int_{\Omega} |K(s, t)|^2 \, d\mu(t) \right) \left( \int_{\Omega} |f(t)|^2 \, d\mu(t) \right) \, d\mu(s) \]

\[ = \int_{\Omega} \int_{\Omega} |K(s, t)|^2 \, d\mu(t) \, d\mu(s) \int_{\Omega} |f(t)|^2 \, d\mu(t) \]

\[ = \int_{\Omega} \int_{\Omega} |K(s, t)|^2 \, d\mu(t) \, d\mu(s) \|f\|^2 \]

part (a) follows.

The range of \( T_{K_n} \) is contained in the linear span of \( a_1, a_2, \ldots, a_n \). This verifies (b).
Let $e_1, e_2, \ldots$ be an orthonormal basis of $L^2(\mu)$. Then $e_i(s)e_j(t)$, $1 \leq i, j$ is an orthonormal basis of $L^2(\mu \times \mu)$. Hence,

$$K(s, t) = \sum_{i,j} k_{i,j} e_i(s)e_j(t)$$

where

$$k_{i,j} = \int_{\Omega} \int_{\Omega} K(s, t)e_i(s)e_j(t) \, d\mu(s) \, d\mu(t)$$

and the series converges in $L^2(\mu \times \mu)$. Let

$$K_{(n)}(s, t) = \sum_{i,j \leq n} k_{i,j} e_i(s)e_j(t).$$

Then $\lim_{n \to \infty} \| K - K_{(n)} \| = 0$. Hence, by (a), $\lim_{n \to \infty} \| T_{K} - T_{K_{(n)}} \| = 0$. Since, by (b), the operators $T_{K_{(n)}}$ have finite dimensional range, we see that $T_{K}$ is compact. This verifies (c).

Let us identify $L^2(\mu)$ with its dual by

$$\langle f, g \rangle = \int_{\Omega} f(s)g(s) \, d\mu(s) \quad (f, g \in L^2(\mu)).$$

Then

$$\langle T_{K}f, g \rangle = \int_{\Omega} T_{K}f(s)g(s) \, d\mu(s) = \int_{\Omega} \int_{\Omega} K(s, t)f(t)g(s) \, d\mu(t) \, d\mu(s)$$

$$= \int_{\Omega} f(t) \int_{\Omega} K(s, t)g(s) \, d\mu(s) \, d\mu(t)$$

$$= \langle f, T_{K^*}g \rangle,$$

where

$$K^*(s, t) = K(t, s).$$