Math. 4103, homework 6, solutions

1. Let $C$ denote the positively oriented boundary of a square whose sides lie along the lines $x = \pm 2, y = \pm 2$. Compute the integral

$$\int_C \frac{\cosh(z)}{z^4} \, dz.$$

Cauchy integral formula implies that

$$\int_C \frac{\cosh(z)}{z^4} \, dz = 2\pi i \frac{1}{3!} \left( \frac{d}{dz} \right)^3 \cosh(z) \big|_{z=0}$$

$$= 2\pi i \frac{1}{3!} \sinh(0) = 0.$$

2. Let $C$ denote the positively oriented circle $|z - i| = 2$. Compute the integral

$$\int_C \frac{1}{(z^2 + 4)^2} \, dz.$$

As in the first problem

$$\int_C \frac{1}{(z^2 + 4)^2} \, dz = \int_C \frac{1}{(z + 2i)^2(z - 2i)^2} \, dz = \int_C \frac{(z + 2i)^{-2}}{(z - 2i)^2} \, dz$$

$$= 2\pi i \left( \frac{d}{dz} \right) \frac{1}{(z + 2i)^2} \big|_{z=2i}$$

$$= -4\pi i (4i)^{-3}$$

3. Suppose $f(z)$ is an entire function (holomorphic on the whole plane) and there is $C < \infty$ such that

$$\text{Re}(f(z)) \leq C \quad (z \in \mathbb{C}).$$

Show that $f$ is a constant function.
Let $F(z) = e^{f(z)}$. Then $|F(z)| = e^{Re(f(z))} \leq e^C$ for all $z \in \mathbb{C}$. Hence, by Liouville’s theorem, $F$ is constant. Therefore

$$0 = F'(z) = f'(z)e^{f(z)} \quad (z \in \mathbb{C}).$$

Thus the derivative of $f$ is zero on the whole plane. This implies that $f$ is a constant.

4. Let $f$ be a continuous function in a closed region $R$. Assume that $f$ is analytic, non-constant and non-zero on $R$. Prove that $|f(z)|$ does not have the minimum in the interior of $R$.

This follows from the Maximum Principle applied to $\frac{1}{f(z)}$.

5. Consider the function $f(z) = (z + 1)^2$ and the closed triangular region with the vertices at $z = 2$, $z = 0$ and $z = i$. Find the points in $R$ where $|f|$ has its minimum and maximum values.

We look for the point $z$ in $R$ such that $|z + 1|^2$ is maximal. Equivalently, we look for the point $z$ in $R$ such that $|z + 1|$ is maximal. Notice that $|z + 1|$ is the distance from $z$ to $-1$. Thus the maximum is at $z = 2$ and the minimum at $z = 0$.


Look it up in the book.