1. Compute the integral

\begin{equation}
\int_0^{2\pi} \cos^3(\theta) \sin(\theta) \, d\theta.
\end{equation}

Let $C$ be the unit circle centered at the origin, oriented in the positive direction. Then the integral (1.1) is equal to

\begin{equation}
\int_C \left( \frac{z + z^{-1}}{2} \right)^3 \left( \frac{z - z^{-1}}{2i} \right) \frac{1}{iz} \, dz.
\end{equation}

Since the constant term of the function

\begin{align*}
(z + z^{-1})^3 (z - z^{-1}) &= (z^3 + 3z^2z^{-1} + 3zz^{-2} + z^{-3})^3 (z - z^{-1})
\end{align*}

is zero, the integral (1.2) is zero.

Here is a straightforward computation:

\[\int_0^{2\pi} \cos^3(\theta) \sin(\theta) \, d\theta = -\frac{1}{4} \cos^4(\theta) \bigg|_0^{2\pi} = -\frac{1}{4} (\cos^4(2\pi) - \cos(0)) = 0.\]

2. Compute the integral

\begin{equation}
\int_0^{2\pi} (\cos^3(\theta) + \sin(\theta)) \, d\theta.
\end{equation}

Let $C$ be the unit circle centered at the origin, oriented in the positive direction. Then the integral (1.1) is equal to

\begin{equation}
\int_C \left( \frac{z + z^{-1}}{2} \right)^3 \left( \frac{z - z^{-1}}{2i} \right) \frac{1}{iz} \, dz.
\end{equation}

Since the constant term of the function

\begin{align*}
\left( \frac{z + z^{-1}}{2} \right)^3 &= \left( \frac{z - z^{-1}}{2i} \right)
\end{align*}

is zero, the integral (1.2) is zero.

Here is a straightforward computation:

\[\int_0^{2\pi} \cos^3(\theta) \sin(\theta) \, d\theta = -\frac{1}{4} \cos^4(\theta) \bigg|_0^{2\pi} = -\frac{1}{4} (\cos^4(2\pi) - \cos(0)) = 0.\]
is zero, the integral (2.2) is zero.

3. Let \( C \) denote the positively oriented boundary of a square whose sides lie along the lines \( x = \pm 2, \ y = \pm 2 \). Compute the integral

\[
\int_C \frac{\cos(z)}{z(z^2 + 8)} \, dz.
\]

We use Cauchy Integral Formula for the function \( \frac{\cos(z)}{z(z^2 + 8)} \), which is holomorphic in a region containing the square:

\[
\int_C \frac{\cos(z)}{z(z^2 + 8)} \, dz = \int_C \frac{\cos(z)(z^2 + 8)^{-1}}{z} \, dz = 2\pi i \cos(0)(0^2 + 8)^{-1} = \frac{\pi i}{4}.
\]

4. Let \( C \) denote the positively oriented boundary of a square whose sides lie along the lines \( x = \pm 2, \ y = \pm 2 \). Compute the integral

\[
\int_C \frac{z}{2z + 1} \, dz.
\]

As in the previous problem,

\[
\int_C \frac{z}{2z + 1} \, dz = \int_C \frac{z^{2-1}}{z - (-1/2)} \, dz = 2\pi i (-1/2)^{2-1} = -\frac{\pi i}{2}.
\]

5. Let \( C \) denote the positively oriented boundary of a square whose sides lie along the lines \( x = \pm 2, \ y = \pm 2 \). Compute the integral

\[
\int_C \frac{z}{(2z + 1)(2z - 1)} \, dz.
\]

The function under the integral has two singular points, \( \pm \frac{1}{2} \), within the square.
Hence, as explained in class, the integral (5.1) is equal to the sum of two integrals: one over a small circle \( C_1 \) around \( \frac{1}{2} \) and the other one over a small circle \( C_2 \) around \( -\frac{1}{2} \). Furthermore,

\[
\int_{C_1} \frac{z}{(2z + 1)(2z - 1)} \, dz = \int_{C_1} \frac{z(2z + 1)^{-1}2^{-1}}{z - 1/2} \, dz = 2\pi i (1/2)(2(1/2) + 1)^{-1}2^{-1} = \frac{\pi i}{4},
\]

and

\[
\int_{C_2} \frac{z}{(2z + 1)(2z - 1)} \, dz = \int_{C_2} \frac{z(2z - 1)^{-1}2^{-1}}{z + 1/2} \, dz = 2\pi i (-1/2)(2(-1/2) - 1)^{-1}2^{-1} = \frac{\pi i}{4}.
\]

Therefore, the integral (5.1) is equal to \( \frac{\pi i}{2} \).

6. Let \( C \) be a simple closed contour described in the positive sense in the \( z \) plane. Define

\[
g(z) = \int_C \frac{s^3 + 2s}{(s - z)^3} \, ds.
\]

Show that \( g(z) = 6\pi iz \) if \( z \) is inside \( C \) and \( g(z) = 0 \) if \( z \) is outside.

If \( z \) is inside \( C \) then \( s \to \frac{s^3 + 2s}{(s - z)^3} \) is holomorphic inside and on \( C \). Hence, Cauchy-Goursat theorem implies that \( g(z) = 0 \).

Suppose \( z \) is inside \( C \). Then, by Cauchy Integral Formula

\[
\frac{2\pi i}{2\pi i} \int_C \frac{s^3 + 2s}{(s - z)^3} \, ds = \left( \frac{d}{ds} \right)^2 (s^3 + 2s) \big|_{s=z} = 6z.
\]

7. Show that if \( f \) is an analytic function within and on a simple closed contour \( C \) and \( z_0 \) is not on \( C \) then

\[
\int_{C} \frac{f'(z)}{z - z_0} \, dz = \int_{C} \frac{f(z)}{(z - z_0)^2} \, dz.
\]
If \( z \) is outside \( C \) then both sides are zero (by Cauchy-Goursat). Otherwise, by Cauchy Integral Formula,

\[
\frac{1}{2\pi i} \int_C \frac{f'(z)}{z - z_0} \, dz = f'(z_0) = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z - z_0)^2} \, dz.
\]

8. Show that for any real number \( a \),

\[
\int_0^\pi e^{a \cos(\theta)} \cos(a \sin(\theta)) \, d\theta = \pi.
\]

By Cauchy Integral Formula,

\[
1 = e^{a0} = \frac{1}{2\pi i} \int_C \frac{e^{az}}{z} \, dz = \frac{1}{2\pi i} \int_{-\pi}^\pi \frac{e^{ae^{i\theta}}}{e^{i\theta}} \, i e^{i\theta} \, d\theta = \frac{1}{2\pi} \int_{-\pi}^\pi e^{a e^{i\theta}} \, d\theta.
\]

Hence,

\[
1 = Re(1) = \frac{1}{2\pi} \int_{-\pi}^\pi Re \left( e^{a e^{i\theta}} \right) \, d\theta = \frac{1}{2\pi} \int_{-\pi}^\pi e^{a \cos(\theta)} \cos(a \sin(\theta)) \, d\theta = \frac{2}{2\pi} \int_0^\pi e^{a \cos(\theta)} \cos(a \sin(\theta)) \, d\theta,
\]

where the last equality follows from the fact that the function under the integral is even.