ODE sec. 5, Spring 2008

Homework 9

1. Find the radius of convergence of the series

\[ \sum_{n=0}^{\infty} \frac{1}{2^n} x^n. \]

Let \( R \) be the radius of convergence of the series. Then

\[ \frac{1}{R} = \lim_{n \to \infty} \left( \frac{1}{2^{n+1}} \right) / \left( \frac{1}{2^n} \right) = \frac{1}{2}. \]

Thus \( R = 2 \).

2. Find the radius of convergence of the series

\[ \sum_{n=0}^{\infty} n^n x^n. \]

Let \( R \) be the radius of convergence of the series. Then

\[ \frac{1}{R} = \lim_{n \to \infty} (n^n)^{\frac{1}{n}} = \lim_{n \to \infty} n = \infty. \]

Thus \( R = 0 \). Hence the series converges for \( x = 0 \) only.

3. Express the series as a combination of elementary functions, such as polynomials, \( e^x, \sin(x) \) or \( \cos(x) \):

\[ \sum_{n=2}^{\infty} \frac{1}{n!} x^n. \]

\[ \sum_{n=2}^{\infty} \frac{1}{n!} x^n = \sum_{n=0}^{\infty} \frac{1}{n!} x^n - \frac{1}{0!} x^0 - \frac{1}{1!} x^1 = \sum_{n=0}^{\infty} \frac{1}{n!} x^n - 1 - x = e^x - 1 - x. \]

4. Express the series as a combination of elementary functions, such as polynomials, \( e^x, \sin(x) \) or \( \cos(x) \):

\[ \sum_{n=3}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}. \]
\[
\sum_{n=3}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n + 1)!} x^{2n+1} - \frac{(-1)^0}{(20 + 1)!} x^{20+1} - \frac{(-1)^1}{(21 + 1)!} x^{21+1} - \frac{(-1)^2}{(22 + 1)!} x^{22+1} = \sin(x) - x + \frac{1}{3} x^3 - \frac{1}{5} x^5.
\]

5. Find the general solution of the equation

\[
(x^2 - 4)y'' = -3xy' - y.
\]