ODE sec. 5, Spring 2008

Homework 5, solutions

1. Find all complex numbers z such that $z^5 = i$.

$$z = e^{k\frac{2\pi i}{5}} e^{\frac{\pi i}{10}} \qquad (k = 0, 1, 2, 3, 4)..$$

Find the general solution of each of the following differential equations.

2. $y'' + 3y' - 10y = 6e^x$.

This is a second order differential equation with constant coefficients. The corresponding homogeneous equation is

$$y'' + 3y' - 10y = 0.$$

The characteristic equation of the homogeneous equation is

$$p^2 + 3p - 10 = 0,$$

which has two solutions $p = -5$ and $p = 2$. Hence the general solution of the homogeneous quation is

$$y_h = Ae^{-5x} + Be^{2x}.$$

We look for a particular solution of the original equation, of the form $y_p = Ce^x$. Then

$$y_p'' + 3y_p' - 10y_p = -6Ce^x.$$

Hence,

$$-6Ce^x = 6e^x.$$

Thus, $C = -1$, $y_p = -e^x$ and therefore the solution of our equation is

$$y = Ae^{-5x} + Be^{2x} - e^x.$$

3. $y'' + 3y' - 10y = 6e^{4x}$.

This is a second order differential equation with constant coefficients. The corresponding homogeneous equation is

$$y'' + 3y' - 10y = 0.$$

The general solution of the homogeneous quation is

$$y_h = Ae^{-5x} + Be^{2x}.$$

We look for a particular solution of the original equation, of the form $y_p = Ce^{4x}$. Then

$$y_p'' + 3y_p' - 10y_p = (16C + 12C - 10C)e^x = 18Ce^{4x}.$$

Hence,

$$18Ce^{4x} = 6e^{4x}.$$

Thus, $C = \frac{1}{3}$, $y_p = \frac{1}{3}e^{4x}$ and therefore the solution of our equation is

$$y = Ae^{-5x} + Be^{2x} + \frac{1}{3}e^{4x}.$$

4. $y'' + y = 2cos(x)$.

The solution of the homogeneous equation

$$y'' + y = 0$$

is

$$y_h = Ae^{ix} + Be^{-ix}.$$

Since

$$cos''(x) + cos(x) = 0,$$

we look for the particular solution of the form

$$y_p = x(acos(x) + bsin(x)).$$

Then

$$\begin{aligned} y_p' &= acos(x) + bsin(x) + x(-asin(x) + bcos(x)), \\ y_p'' &= 2(-asin(x) + bcos(x)) + x(-acos(x) - bsin(x)), \\ y_p'' + y_p &= 2(-asin(x) + bcos(x)). \end{aligned}$$

Thus

$$2(-asin(x) + bcos(x)) = 2cos(x).$$

Hence, $a = 0$ and $b = 1$. Therefore $y_p = xsin(x)$. The general solution of the original equation is

$$y = Ae^{ix} + Be^{-ix} + xsin(x).$$

5. $y'' + 2y' + y = e^{-x}ln(x)$.

We use the method of variation of prameters. The two independent solutions of the homogeneous equation

$$y'' + 2y' + y = 0$$

are

$$y_1 = e^{-x}, \quad y_2 = xe^{-x}.$$

Hence the system of two linear equations for v_1' and v_2' is

$$\begin{aligned} v_1'\ e^{-x} + v_2'xe^{-x} &= 0 \\ -v_1'\ e^{-x} + v_2'(e^{-x} - xe^{-x}) &= e^{-x}ln(x). \end{aligned}$$

This system has solutions

$$v_1' = -xln(x), \quad v_2' = ln(x).$$

Hence,

$$v_1 = -\frac{1}{2}x^2 ln(x) + \frac{1}{4}x^2, \quad v_2 = xln(x) - x.$$

Therefore the general solution we are looking for is

$$y = Ae^{-x} + Bxe^{-x} + (-\frac{1}{2}x^2 ln(x) + \frac{1}{4}x^2)e^{-x} + (xln(x) - x)xe^{-x}$$

6. $y'' - 2y' - 3y = 64xe^{-x}$.

The solution of the homogeneous equation is

$$y_h = Ae^{-x} + Be^{3x}.$$

We look for the particular solution of the form $y_p = x(ax+b)e^{-x}$. Then

$$\begin{aligned} y_p' &= (-ax^2 + (2a-b)x + b)e^{-x} \\ y_p'' &= (ax^2 - (4a-b)x + 2a - 2b)e^{-x}, \end{aligned}$$

so that

$$y_p'' - 2y_p' - 3y_p = (-8ax + 2a - 4b)e^{-x}.$$

Thus

$$64 = -8a, \quad 0 = a - 2b.$$

Hence, $a = -8$ and $b = -4$. Thus $y_p = (-8x - 4)e^{-x}$. Therefore the general solution of the original equation is

$$y = Ae^{-x} + Be^{3x} - (8x^2 + 4x)e^{-x}.$$

7. $y'' - 4y' + 4y = 0$.

This is a homogeneous second order equation with constant coefficients. The characteristic equation is

$$p^2 - 4p + 4 = 0.$$

Hence $p = 2$ and the solution is

$$y = Ae^{2x} + Bxe^{2x}.$$

8. $xy'' + y' = 4x$.

We use the substitution $z = y'$, $z' = y''$ and divide both sides by x. Then

$$z' + x^{-1}z = 4.$$

This is a first order linear equation with $a(x) = x^{-1}$. Since,

$$e^{\int a(x)\,dx} = x.$$

our equation transforms to

$$(xz)' = 4x.$$

Therefore

$$xz = 2x^2 + A$$

Thus

$$y' = 2x + Ax^{-1}.$$

We integrate and get the general solution

$$y = x^2 + A\,ln(x) + B.$$

9. $y'' = 1 + (y')^2$.

We reduce the order via the substitution $z = y'$, $z\frac{dz}{dy} = y''$. This leads to the following separable equation

$$z\frac{dz}{dy} = 1 + z^2.$$

We separate the variables, integrate and solve for z:

$$\begin{aligned} \frac{z}{z^2+1}dz &= dy \\ ln(z^2+1) &= 2y + A \\ z &= (e^A e^{2y} - 1)^{1/2}. \end{aligned}$$

Hence,

$$\int (e^A e^{2y} - 1)^{-1/2} dy = \int dx = x + C.$$

We compute the integral

$$\int (e^A e^{2y} - 1)^{-1/2} dy = \int e^{-A/2} e^{-y} (1 - e^{-A} e^{-2y})^{-1/2} dy$$

using the substitution $u = e^{-A/2}e^{-y}$. Then the last integral is

$$-\int (1 - u^2)^{-1/2}\, du.$$

Let $u = cos(\theta)$. Then the last integral is

$$\int d\theta = \theta = cos^{-1}(u).$$

Hence,

$$\begin{aligned} cos^{-1}(u) &= x + C \\ u &= cos(x + C) \\ e^{-A/2}e^{-y} &= cos(x + C) \\ y &= -ln(cos(x + C)) + D. \end{aligned}$$

10. $xy' = y + 2xe^{-y/x}$.

Since this is a homogeneous equation,

$$y' = y/x + 2e^{-y/x},$$

we use the substitution $z = y/x$ and obtain the following equation for z:

$$xz' = 2e^{-z}.$$

This is a separable equation with the solution

$$z = ln(2ln(x) + A).$$

Hence,

$$y = x\,ln(2\,ln(x) + A).$$

11. $(x^3 + y^3)dx - xy^2dy = 0$.

Since this is a homogeneous equation,

$$y' = \frac{x^3 + y^3}{xy^2} = \frac{1 + (y/x)^3}{(y/x)^2},$$

we use the substitution $z = y/x$ and obtain the following equation for z:

$$xz' = z^{-2}.$$

This is a separable equation with the solution

$$\frac{1}{3}z^3 = ln(x) + A.$$

Hence,

$$y^3 = 3x^3(ln(x) + A).$$

12. $(y + ycos(xy))dx + (x + xcos(xy))dy = 0.$

This is an exact equation and we compute

$$f = \int (y + ycos(xy))dx = xy + sin(xy) + \phi(y),$$

$$x + xcos(xy) = \partial_y f = x + xcos(xy) + \phi'(y).$$

Hence, $\phi'(y) = 0$ so that ϕ is a constant. Thus the solution is

$$xy + sin(xy) = C.$$

13. $(sin(x)sin(y) - xe^y)dy = (e^y + cos(x)cos(y))dx.$

This is an exact equation

$$(e^y + cos(x)cos(y))dx - (sin(x)sin(y) - xe^y)dy = 0,$$

and we compute

$$f = \int (e^y + cos(x)cos(y))dx = xe^y + sin(x)cos(y) + \phi(y),$$

$$-(sin(x)sin(y) - xe^y) = \partial_y f = -(sin(x)sin(y) - xe^y) + \phi'(y).$$

Hence, $\phi'(y) = 0$ so that ϕ is a constant. Thus the solution is

$$xe^y + sin(x)cos(y) = C.$$

14. $y' + 2xy = 0.$

This is a separable equation with the solution

$$y = Ae^{-x^2}.$$

15. $y' + y = 2xe^{-x} + x^2.$

This is a first order linear equation with the solution

$$y = (x^2 + C)e^{-x} + x^2 - 2x + 2.$$