ODE sec. 5, Spring 2008

Homework 3, solutions

1. Recall the definition of the gradient of a function of two variables: $f(x,y)$ (i.e. copy it from a book).

Let $f(x,y)$ be a differentiable function of two real variables, defined on some open subset of the plane. Then the gradient of f is defined as

$$grad(f)(x,y) = (\partial_x f(x,y), \partial_y f(x,y)) \,.$$

2. Recall the definition of the level curve of a function of two variables: $f(x,y)$ (i.e. copy it from a book).

A level curve of a function $f(x,y)$ of two real variables is defined as the follwing subset of the plane,

$$\{(x,y);\ f(x,y) = c\},$$

where c is a constant.

Solve the following differential equations:

3. $x^2 y' + y = x^2$.

This is a first order linear equation:

$$y' + x^{-2} y = 1.$$

Since

$$\int x^{-2}\, dx = -x^{-1},$$

we multiply both sides by $e^{-x^{-1}}$ and see that

$$\left(e^{-x^{-1}} y\right)' = e^{-x^{-1}}.$$

Therefore,

$$y = e^{x^{-1}} \int e^{-x^{-1}}\, dx.$$

The above integral is impossible to compute explicitly, thus we may stop here. We might approximate the integral via a power series as follows. Since,

$$e^z = \sum_{n=0}^{\infty} \frac{1}{n!} z^n,$$

we see that

$$e^{-x^{-1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} x^{-n} = 1 - x^{-1} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} x^{-n}.$$

Hence

$$\int e^{-x^{-1}}\, dx = x - ln(x) + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \frac{1}{-n+1} x^{-n+1} + C,$$

where C is a constant.

4. $y' = (x^2 + y^2)(x^2 - y^2)^{-1}$.

This is a homogeneous equation

$$y' = \frac{1+z^2}{1-z^2}, \quad z = \frac{y}{x}.$$

Hence, $y' = z + xz'$, so that

$$z + xz' = \frac{1+z^2}{1-z^2}.$$

We simplify:

$$xz' = \frac{1+z^2-z+z^3}{1-z^2}.$$

This is a separable equation:

$$\frac{1-z^2}{z^3+z^2-z+1}\, dz = \frac{dx}{x}.$$

Hence,

$$\int \frac{1-z^2}{z^3+z^2-z+1}\, dz = ln(x) + A,$$

where A is a constant. The integral on the left hand side is hard to compute because the equation $z^3 + z^2 - z + 1 = 0$ does not have rational solutions. We may look for approximations via a computer, or stop here.

5. $-sin(x)sin(y)\, dx + cos(x)cos(y)\, dy = 0$.

This is an exact equation. We look for a function f such that $\partial_x f = -sin(x)sin(y)$ and $\partial_y f = cos(x)cos(y)$. The usual method gives $f(x, y) = cox(x)sin(y)$. Thus the solutions of the differential equation are the level curves:

$$cox(x)sin(y) = C.$$

6. $y\, y'' - (y')^2 = 0$.

We reduce the order of the equation by the substitution: $z = y'$ and $y'' = z\frac{dz}{dy}$. Then the equation is transformed to

$$yz\frac{dz}{dy} = z^2.$$

This is a separable equation

$$\frac{dz}{z} = \frac{dy}{y}.$$

We integrate and get

$$ln(|z|) = ln(|y|) + C.$$

Hence,

$$z = Ay.$$

Therefore

$$\frac{dy}{dx} = Ay.$$

Again, this is a separable equation

$$\frac{dy}{y} = Adx.$$

We integrate both sides:

$$y = Be^{Ax},$$

where A and B are arbitrary constants.

5. $y' + y = 2xe^{-x} + x^2$.

This is a first order linear equation with the solution

$$y = (x^2 + C)e^{-x} + x^2 - 2x + 2.$$

7. $x\,y'' - 3y' = 5x$.

We reduce the order of the equation by the substitution: $z = y'$. Then the equation is transformed to

$$z' - 3x^{-1}z = 5.$$

This is a first order linear equation. Since

$$\int -3x^{-1}\,dx = ln(x^{-3}),$$

we multiply both sides by x^{-3} and see that

$$\left(x^{-3}z\right)' = 5x^{-3}.$$

Therefore,

$$x^{-3}z = -\frac{5}{2}x^{-2} + A,$$

so that

$$z = -\frac{5}{2}x + Ax^3.$$

Hence, via an integration,

$$y = -\frac{5}{4}x^2 + Bx^4 + C,$$

where B and C are arbitrary constants.

8. Find the orthogonal trajectories to the family of curves $y = c(x^2 + 1)$.

By taking the derivative with respect to x we get the following equation

$$y' = 2cx.$$

Since $c = y(x^2 + 1)^{-1}$, we may eliminate this constant:

$$y' = \frac{2xy}{x^2 + 1}.$$

The orthogonal family satisfies the following equation

$$y' = -\frac{x^2 + 1}{2xy},$$

which is separable:

$$-2y\,dy = \frac{x^2 + 1}{x}\,dx.$$

We integrate both sides and conclude that

$$-y^2 = ln(x) + \frac{1}{2}x^2 + C.$$

The last equation describes the orthogonal family we have been looking for.