ODE sec. 5, Spring 2008

Homework 2, solutions

1. State the Fundamental Theorem of Calculus (i.e. copy it from a book).

Let f be a continuous function defined on a closed interval $[a, b]$, with values in the real numbers. Then the formula

$$\int_a^x f(t)\,dt \qquad (a < x < b)$$

defines a differentiable function of x and

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x) \qquad (a < x < b).$$

2. Sate the following basic properties of the exponential function:
(a) $e^{a+b} = e^a e^b$,
(b) $\frac{d}{dx}e^x = e^x$,

3. Sate the following basic properties of the natural logarithm:
(a) $ln(ab) = ln(a) + ln(b)$,
(b) $\frac{d}{dx}ln(x) = x^{-1}$,
(c) $\int \frac{1}{x}\,dx = ln(|x|)$.

4. State the Chain Rule (i.e. copy it from a book).

$$\frac{d}{dx}f(g(x)) = f'(g(x))g'(x).$$

5. Determine if the following equation is exact and solve:

$$(x + 2y^{-1})dy + ydx = 0.$$

Here $M(x, y) = y$, $N(x, y) = x + 2y^{-1}$ and $\partial_y M = 1 = \partial_x N$. Thus the equation is exact.

We look for a function $f(x, y)$ such that

$$\partial_x f = M = y.$$

Thus

$$f(x, y) = xy + \phi(y).$$

On the other hand

$$x + 2y^{-1} = \partial_y f = x + \phi'(y).$$

Therefore

$$\phi'(y) = 2y^{-1},$$

so that

$$\phi(y) = ln(y^2).$$

Hence, $f(x,y) = xy + ln(y^2)$ and the solution is given by the equation

$$xy + ln(y^2) = C,$$

where C is an arbitrary constant.

6. Determine if the following equation is exact and solve:

$$\frac{xdx}{(x^2+y^2)^{3/2}} + \frac{ydy}{(x^2+y^2)^{3/2}} = 0.$$

Here $M(x,y) = \frac{x}{(x^2+y^2)^{3/2}}$, $N(x,y) = \frac{y}{(x^2+y^2)^{3/2}}$

$$\partial_y M = x(-3/2)(x^2+y^2)^{-5/2}2y, \text{ and } \partial_x N = y(-3/2)(x^2+y^2)^{-5/2}2x.$$

Thus $\partial_y M = \partial_x N$ so that the equation is exact.

We look for a function $f(x,y)$ such that

$$\partial_x f = M = \frac{x}{(x^2+y^2)^{3/2}}.$$

Thus

$$f(x,y) = -(x^2+y^2)^{-1/2} + \phi(y).$$

On the other hand

$$\frac{y}{(x^2+y^2)^{3/2}} = \partial_y f = \frac{y}{(x^2+y^2)^{3/2}} + \phi'(y).$$

Therefore

$$\phi'(y) = 0,$$

so that

$$\phi(y) = A,$$

and we may assume that $A = 0$. Hence, $f(x,y) = -(x^2+y^2)^{-1/2}$ and the solution is given by the equation

$$-(x^2+y^2)^{-1/2} = C,$$

where C is an arbitrary negative constant. The last equation may be rewritten as

$$x^2 + y^2 = \left(\frac{1}{-C}\right)^2.$$

Thus the solution curves are circles centered at the origin.

7. Sketch the following family of curves, find the orthogonal family add those to your sketch:

$$y = c(1 + cos(x)).$$

This family satisfies the following differential equation

$$y' = -csin(x) = -\frac{y}{1 + cos(x)} sin(x).$$

Replacing y' by $-\frac{1}{y'}$ we obtain the following equation

$$-\frac{1}{y'} = -\frac{y}{1 + cos(x)} sin(x),$$

or equivalently

$$y' = \frac{1 + cos(x)}{ysin(x)}.$$

This is a separable equation

$$ydy = \frac{1 + cos(x)}{sin(x)} dx.$$

We integrate and get

$$\begin{aligned} \frac{1}{2}y^2 = & \int csc(x)\, dx + \int cot(x)\, dx \\ = & ln(|csc(x) - cot(x)|) + ln(|sin(x)|) + A. \end{aligned}$$

Hence,

$$y^2 = ln((csc(x) - cot(x))^2) + ln(sin^2(x)) + B,$$

where $B = 2A$ is an arbitrary constant. This is the equation describing the orthogonal family of curves.

8. Sketch the following family of curves, find the orthogonal family add those to your sketch:

$$x + y = c.$$

This family satisfies the following differential equation

$$1 + y' = 0.$$

Replacing y' by $-\frac{1}{y'}$ we obtain the following equation

$$1 - \frac{1}{y'} = 0.$$

Equivalently,

$$y' = 1.$$

The solution of this last equation is

$$y = x + C.$$

This is the equation describing the orthogonal family of curves (in this case straight lines).

9. Verify that the following equation is homogeneous and solve it:

$$(x^2 - 2y^2)dx + xydy = 0.$$

We solve for $y' = \frac{dy}{dx}$:

$$y' = \frac{2y^2 - x^2}{xy} = 2\frac{y}{x} - \frac{x}{y}.$$

We see that this is a homogeneous equation and use the substitution $z = y/x$ to solve it.

$$\begin{aligned} y' =& xz' + z, \\ xz' + z =& 2z - z^{-1}, \\ xz' =& z - z^{-1} = \frac{z^2 - 1}{z}. \end{aligned}$$

The last equation is separable:

$$\frac{z}{z^2 - 1}\, dz = \frac{dx}{x}.$$

Hence, via an integration

$$ln(|z^2 - 1|) = ln(x^2) + A.$$

Therefore

$$z^2 - 1 = Bx^2.$$

Hence

$$y^2 = Bx^4 + x^2.$$

10. Solve the following differential equation:

$$yy'' - (y')^2 = 0.$$

Here we reduce the order of the equation via the substitution $z = y'$, so that $y'' = z\frac{dz}{dy}$. This leads to the following equation

$$yz\frac{dz}{dy} = z^2.$$

This is a separable equation:

$$\frac{dz}{z} = \frac{dy}{y}.$$

Hence,

$$z = Ay.$$

But $z = y'$. We integrate and get

$$y = Be^{Ax},$$

where A and B are arbitrary constants.