ODE sec. 5, Spring 2008

Final Exam, solutions

Name (please print):

Student No.:

1. Perform Picard's algorithm to find the approximations $y_0(x)$, $y_1(x)$ and $y_2(x)$, for the initial value problem: $\frac{dy}{dx} = e^x - y^2$, $y(0) = 0$.

$$y_0(x) = 0.$$

$$y_1(x) = \int_0^x e^t\,dt = e^x - 1.$$

$$y_2(x) = \int_0^x (e^t - (e^t - 1)^2)\,dt = -\frac{1}{2}e^{2x} + 3e^x - x - \frac{5}{2}.$$

2. Find a general solution of the equation: $(y^2 - x)\frac{dy}{dx} - y + x^2 = 0$.

This is an exact equation and the solution is given implicitly by

$$\frac{1}{3}x^3 + \frac{1}{3}y^3 - xy = C.$$

3. Solve the initial value problem: $y'' + 5y' + 4y = 0$, $y(0) = 1$, $y'(0) = 1$.

This is a homogeneous second order equation. The characteristic equation looks as follows:

$$0 = r^2 + 5r + 4 = (r+1)(r+4).$$

Hence, the general solution is

$$y = Ae^{-x} + Be^{-4x}.$$

From the initial condition we find $A$ and $B$. Thus the solution is

$$y = \frac{5}{3}e^{-x} - \frac{2}{3}e^{-4x}.$$

4. Find the approximation $a_0 + a_1x + a_2x^2$ of the solution $y = a_0 + a_1x + a_2x^2 + ...$ of the following initial value problem:

$$y'' + xy' - y = 0,\ y(0) = 0,\ y'(0) = 1.$$

The approximate solution is $y = x$.

5. Solve the initial value problem.

$$\begin{aligned} x_1'(t) =& -x_1(t) + 2x_2(t) \\ x_2'(t) =& 3x_1(t) + 4x_2(t) \\ x_1(0) =& 2, \\ x_2(0) =& 2. \end{aligned}$$

The eigenvalues of the matrix of this equation are 5 and $-1$. We compute the corresponding eigenvextors and obtain the general solution

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = C_1 e^{5t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + C_2 e^{-2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

We find the constants from the initial condition and obtain

$$\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \frac{6}{7} e^{5t} \begin{pmatrix} 1 \\ 3 \end{pmatrix} + \frac{4}{7} e^{-2t} \begin{pmatrix} 2 \\ -1 \end{pmatrix}.$$

6. Find the general solution of the following equation,

$$(x^2 - 2y^2)dx + xydy = 0.$$

This is a homogeneous equation. The substitution $y = vx$ leads to the solution for $V$:

$$\frac{1}{2} ln(|v^2 - 1|) = ln(|x|) + C.$$

This is equivalent to

$$y^2 = Ax^4 + x^2.$$

7. Find the general solution of the following equation,

$$xy' + y = x^2.$$

This is a first order linear equation:

$$y' + x^{-1}y = x,$$

and the solution is

$$y = \frac{x^3}{3} + \frac{A}{x}.$$

8. Draw the field of slopes for the differential equation $y' = x$.

The slope at a point $(x, y)$ is $x$.