ODE sec. 5, Spring 2008,

Name:

Exam 2, solutions

1. Solve the initial value problem,

$$
\begin{aligned}
x_1'(t) =& - x_1(t) + 2x_2(t)\\
x_2'(t) =& 3x_1(t) + x_2(t)\\
x_1(0) =& 2,\\
x_2(0) =& 1.
\end{aligned}
$$

First we use the Laplace transform.
Let $f_k = f_k(s)$ be the Laplace transform of x_k, $k = 1, 2$. Then our system of equations becomes

$$
\begin{aligned}
sf_1 - 2 &= -f_1 + 2f_2\\
sf_2 - 1 &= 3f_1 + f_2.
\end{aligned}
\tag{1.1}
$$

In matrix form (1.1) may be rewritten as

$$\begin{pmatrix} s+1 & -2 \\ -3 & s-1 \end{pmatrix} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

Hence,

$$
\begin{aligned}
\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} =& \begin{pmatrix} s+1 & -2 \\ -3 & s-1 \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\\
=& \frac{1}{(s+1)(s-1)-6} \begin{pmatrix} s-1 & 2 \\ 3 & s+1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \end{pmatrix}\\
=& \frac{1}{s^2-7} \begin{pmatrix} 2s \\ s+7 \end{pmatrix}
\end{aligned}
$$

Thus

$$f_1 = \frac{2s}{(s-\sqrt{7})(s+\sqrt{7})} = \frac{1}{s-\sqrt{7}} + \frac{1}{s+\sqrt{7}},$$

and

$$f_2 = \frac{s+7}{(s-\sqrt{7})(s+\sqrt{7})} = \frac{1+\sqrt{7}}{2(s-\sqrt{7})} + \frac{1-\sqrt{7}}{2(s+\sqrt{7})}.$$

Therefore,

$$
\begin{aligned}
x_1(t) =& e^{\sqrt{7}t} + e^{-\sqrt{7}t}, \text{ and}\\
x_2(t) =& \frac{1+\sqrt{7}}{2} e^{\sqrt{7}t} + \frac{1-\sqrt{7}}{2} e^{-\sqrt{7}t}.
\end{aligned}
\tag{2.1}
$$

Alternatively, we use the eigenvalue method to arrive at the same solution (2.1) as follows.
The matrix associated to our system is

$$A = \begin{pmatrix} -1 & 2 \\ 3 & 1 \end{pmatrix}.$$

The characteristic equation

$$0 = det(\lambda I - A) = \lambda^2 - 7$$

has two solutions $\lambda = \pm\sqrt{7}$. The corresponding eigenvectors are

$$v_1 = \begin{pmatrix} 1 \\ \frac{1+\sqrt{7}}{2} \end{pmatrix}, \quad v_2 = \begin{pmatrix} 1 \\ \frac{1-\sqrt{7}}{2} \end{pmatrix}.$$

Hence, the general solution is

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 e^{\sqrt{7}t} \begin{pmatrix} 1 \\ \frac{1+\sqrt{7}}{2} \end{pmatrix} + C_2 e^{-\sqrt{7}t} \begin{pmatrix} 1 \\ \frac{1-\sqrt{7}}{2} \end{pmatrix}.$$

The initial condition reads

$$\begin{pmatrix} 2 \\ 1 \end{pmatrix} = C_1 \begin{pmatrix} 1 \\ \frac{1+\sqrt{7}}{2} \end{pmatrix} + C_2 \begin{pmatrix} 1 \\ \frac{1-\sqrt{7}}{2} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{7}}{2} & \frac{1-\sqrt{7}}{2} \end{pmatrix} \begin{pmatrix} C_1 \\ C_2 \end{pmatrix}$$

Hence,

$$\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ \frac{1+\sqrt{7}}{2} & \frac{1-\sqrt{7}}{2} \end{pmatrix}^{-1} \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Thus,

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = e^{\sqrt{7}t} \begin{pmatrix} 1 \\ \frac{1+\sqrt{7}}{2} \end{pmatrix} + e^{-\sqrt{7}t} \begin{pmatrix} 1 \\ \frac{1-\sqrt{7}}{2} \end{pmatrix}, \tag{2.2}$$

which coincides with (2.1).

2. In terms of Picard's algorithm, compute the successive approximations y_0, y_1, y_2 of the solution y of the given initial value problem,

$$\frac{dy}{dx} = x + y^2, \ y(0) = 1.$$

Here, $y_0(x) = 1$,

$$y_1(x) = 1 + \int_0^x (t + 1^2)\, dt = \frac{1}{2}x^2 + x + 1,$$

and

$$
\begin{aligned}
y_2(x) =& 1 + \int_0^x (t + (\frac{1}{2}t^2 + t + 1)^2)\,dt \\
=& 1 + \int_0^x (t + \frac{1}{4}t^4 + t^2 + 1 + t^3 + t^2 + 2t)\,dt \\
=& \frac{1}{20}x^5 + \frac{1}{4}x^4 + \frac{2}{3}x^3 + \frac{3}{2}x^2 + x + 1.
\end{aligned}
$$

3. Solve the following initial value problem:

$$y''(t) + y(t) = te^t, \quad y(0) = 1, \quad y'(0) = 0.$$

Let Y be the Laplace transform of y. Then

$$s^2Y - s + Y = \frac{1}{(s-1)^2}.$$

Hence,

$$Y = \frac{s}{s^2+1} + \frac{1}{(s-1)^2(s^2+1)}. \tag{3.1}$$

Next we decompose the second summand in (3.1) into partial fractions

$$\frac{1}{(s-1)^2(s^2+1)} = \frac{a}{(s-1)} + \frac{b}{(s-1)^2} + \frac{cs+d}{(s^2+1)}.$$

Here,

$$b = \frac{1}{(s^2+1)}|_{s=1} = \frac{1}{2}.$$

Hence,

$$\frac{1}{(s-1)^2(s^2+1)} - \frac{1}{2(s-1)^2} = \frac{a}{(s-1)} + \frac{cs+d}{(s^2+1)}.$$

Which simplifies to

$$\frac{-s-1}{2(s-1)(s^2+1)} = \frac{a}{(s-1)} + \frac{cs+d}{(s^2+1)}.$$

Hence,

$$a = \frac{-s-1}{2(s^2+1)}|_{s=1} = -\frac{1}{2}.$$

Thus,

$$\frac{-s-1}{2(s-1)(s^2+1)} + \frac{1}{2(s-1)} = \frac{cs+d}{(s^2+1)}.$$

This simplifies to

$$\frac{s}{2(s^2+1)} = \frac{cs+d}{(s^2+1)}.$$

Thus,

$$\begin{aligned} Y =& \frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2} + \frac{1}{2}\frac{s}{(s^2+1} \\ =& \frac{3}{2}\frac{s}{s^2+1} - \frac{1}{2}\frac{1}{s-1} + \frac{1}{2}\frac{1}{(s-1)^2}. \end{aligned}$$

Therefore,

$$y(t) = \frac{3}{2}cos(t) - \frac{1}{2}e^t + \frac{1}{2}te^t. \tag{3.3}$$

4. Find the approximation $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3$ of the general solution $y(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + ... + a_nx^n + ...$ of the initial value problem:

$$(x^2-1)y'' = -xy' - y,\ y(0) = 1,\ y'(0) = 2.$$

We shall keep track of the terms $a_0 + a_1x + a_2x^2 + a_3x^3$ only. Thus

$$\begin{aligned} y =& a_0 + a_1x + a_2x^2 + a_3x^3 + ... \\ y' =& a_1 + 2a_2x + 3a_3x^2 + ... \\ y'' =& 2a_2 + 6a_3x + ... \\ x^2y'' =& 2a_2x^2 + 6a_3x^3 + ... \\ xy' =& a_1x + 2a_2x^2 + 3a_3x^3 + ... \end{aligned}$$

Hence,

$$\begin{aligned} x^2y'' - y'' =& -2a_2 - 6a_3x + 2a_2x^2 + 6a_3x^3 + ... \\ -xy' - y =& -a_0 - 2a_1x - 3a_2x^2 - 4a_3x^3 + ... \end{aligned}$$

Therefore,

$$a_0 = 2a_2,\ 2a_1 = 6a_3.$$

Since, $a_0 = y(0) = 1$ and $a_1 = y'(0) = 2$, we see that

$$a_2 = \frac{1}{2},\ a_3 = \frac{2}{3}.$$

Hence, our approximate solution is

$$1 + 2x + \frac{1}{2}x^2 + \frac{2}{3}x^3. \tag{4.1}$$