We verified the following theorem in class.

**Theorem.** Let $f$ be a continuous, real valued, function defined on a rectangle $(a, b) \times (c, d)$. Assume that there are constants $M$ and $C$ such that for any $(x, y)$ in the rectangle
\[ |f(x, y)| \leq M \quad \text{and} \quad |\partial_y f(x, y)| \leq C. \]
Fix a point $(x_0, y_0)$ in the rectangle. Let $\alpha = \min\{|a - x_0|, |b - x_0|\}$ and let $\beta = \min\{|c - y_0|, |d - y_0|\}$. Let $h$ be any positive number such that
\[ h < \min\{\alpha, \frac{\beta}{M + \beta C}\}. \]
Then Picard’s algorithm
\[ y_0(x) = y_0, \]
\[ y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t)) \, dt \quad (|x - x_0| < h, \ n = 0, 1, 2, \ldots) \]
provides a sequence of functions $y_n$ which converges to a function
\[ y(x) = \lim_{n \to \infty} y_n(x) \quad (|x - x_0| < h), \]
such that
\[ y'(x) = f(x, y(x)), \ y(x_0) = y_0, \quad (|x - x_0| < h). \]
Moreover,
\[ |y(x) - y_L(x)| \leq Mh \frac{(Ch)^L}{1 - Ch} \quad (|x - x_0| < h). \]
Furthermore, if $\tilde{y}(x)$, $|x - x_0| < h$, is another function such that
\[ \tilde{y}'(x) = f(x, \tilde{y}(x)), \ \tilde{y}(x_0) = y_0, \quad (|x - x_0| < h), \]
then $y = \tilde{y}$.

Find the solution of each of the following integral equation equations.

1. $y(x) = 2 + \int_{1}^{x} (t + y(t)) \, dt$.

The equivalent initial value problem is
\[ y' = x + y, \ y(1) = 2. \]

The linear equation to be solved is
\[ y' - y = x. \]
We multiply both sides by $e^{-x}$ and obtain

$$(e^{-x}y)' = xe^{-x}.$$ 

After integration

$$e^{-x}y = -xe^{-x} - e^{-x} + C.$$ 

Hence,

$$y = -x - 1 + Ce^x.$$ 

The constant $C$ is determined from the initial condition

$$2 = -1 - 1 + Ce.$$ 

Thus

$$y = -x - 1 + \frac{4}{e}e^x.$$ 

2. $y(x) = \int_0^x 2t(1 + y(t)) \, dt.$

The equivalent initial value problem is

$$y' = 2x(y + 1), \quad y(0) = 0.$$ 

The separable equation to be solved is

$$y' = 2x(y + 1).$$ 

We multiply both sides by $dx$, divide by $y + 1$ and get

$$\frac{dy}{y + 1} = 2x \, dx.$$ 

After integration

$$y + 1 = Ae^{x^2}.$$ 

The constant $C$ is determined from the initial condition

$$0 + 1 = Ae^0.$$ 

Thus

$$y = e^{x^2} - 1.$$ 

In the following exercises compute the successive approximations $y_1$, $y_2$, $y_3$ of the solution $y$ of the given initial value problem.

3. $\frac{dy}{dx} = -2y, \quad y(0) = 4.$
Here \( f(x, y) = -2y \), \( x_0 = 0 \) and \( y_0 = 4 \). Hence,

\[
y_1(x) = 4 + \int_0^x (-8) \, dt = -8x + 4;
\]
\[
y_2(x) = 4 + \int_0^x (16t - 8) \, dt = 8x^2 - 8x + 4;
\]
\[
y_3(x) = 4 + \int_0^x (-16t^2 + 16t - 8) \, dt = -\frac{16}{3}x^3 + 8x^2 - 8x + 4.
\]

4. \( \frac{dy}{dx} = 3x^2y, \ y(0) = 2 \).

Here \( f(x, y) = 3x^2y \), \( x_0 = 0 \) and \( y_0 = 2 \). Hence,

\[
y_1(x) = 2 + \int_0^x 3t^2 \, dt = 2x^3 + 2;
\]
\[
y_2(x) = 2 + \int_0^x 3t^2(2t^3 + 2) \, dt = x^6 + 2x^3 + 2;
\]
\[
y_3(x) = 2 + \int_0^x 3t^2(t^6 + 2t^3 + 2) \, dt = \frac{1}{3}x^9 + x^6 + 2x^3 + 2.
\]

5. \( \frac{dy}{dx} = y + e^x, \ y(0) = 0 \).

Here \( f(x, y) = y + e^x \), \( x_0 = 0 \) and \( y_0 = 0 \). Hence,

\[
y_1(x) = \int_0^x e^t \, dt = e^x - 1;
\]
\[
y_2(x) = \int_0^x (2e^t - 1) \, dt = 2e^x - x - 2;
\]
\[
y_3(x) = \int_0^x (3e^t - t - 2) \, dt = 3e^x - \frac{1}{2}x^2 - 2x - 3.
\]

6. \( \frac{dy}{dx} = -2xy, \ y(0) = 1 \).

Here \( f(x, y) = -2xy \), \( x_0 = 0 \) and \( y_0 = 1 \). Hence,

\[
y_1(x) = 1 + \int_0^x (-2t) \, dt = -x^2 + 1;
\]
\[
y_2(x) = 1 + \int_0^x (-2t)(-t^2 + 1) \, dt = \frac{1}{2}x^4 - x^2 + 1;
\]
\[ y_3(x) = 1 + \int_0^x (-2t)(\frac{1}{2}t^4 - t^2 + 1) \, dt = \frac{-1}{6}x^6 + \frac{1}{2}x^4 - x^2 + 1. \]

7. \( \frac{dy}{dx} = y^2, \ y(0) = 1. \)

Here \( f(x, y) = y^2, \ x_0 = 0 \) and \( y_0 = 1. \) Hence,

\[ y_1(x) = 1 + \int_0^x dt = x + 1; \]

\[ y_2(x) = 1 + \int_0^x (t + 1)^2 \, dt = \frac{1}{3}x^3 + x^2 + x + 1; \]

\[ y_3(x) = 1 + \int_0^x \left(\frac{1}{3}t^3 + t^2 + t + 1\right)^2 \, dt = \frac{1}{63}x^7 + \frac{1}{9}x^6 + \frac{1}{3}x^5 + \frac{2}{3}x^4 + x^3 + x^2 + x + 1. \]