We verified the following theorem in class.

**Theorem.** Let $f$ be a continuous, real valued, function defined on a rectangle $(a, b) \times (c, d)$. Assume that there are constants $M$ and $C$ such that for any $(x, y)$ in the rectangle

$$|f(x, y)| \leq M \quad \text{and} \quad |\partial_y f(x, y)| \leq C.$$

Fix a point $(x_0, y_0)$ in the rectangle. Let $\alpha = \min\{|a - x_0|, |b - x_0|\}$ and let $\beta = \min\{|c - y_0|, |d - y_0|\}$. Let $h$ be any positive number such that

$$h < \min\{\alpha, \frac{\beta}{M + \beta C}\}.$$

Then Picard’s algorithm

$$y_0(x) = y_0,$$

$$y_{n+1}(x) = y_0 + \int_{x_0}^{x} f(t, y_n(t)) \, dt \quad (|x - x_0| < h, \ n = 0, 1, 2, \ldots)$$

provides a sequence of functions $y_n$ which converges to a function

$$y(x) = \lim_{n \to \infty} y_n(x) \quad (|x - x_0| < h),$$

such that

$$y'(x) = f(x, y(x)), \ y(x_0) = y_0, \quad (|x - x_0| < h).$$

Moreover,

$$|y(x) - y_L(x)| \leq Mh \frac{(Ch)^L}{1 - Ch} \quad (|x - x_0| < h).$$

Find the general solution of each of the following integral equation equations.

1. $y(x) = 1 + \int_{1}^{x} (2t \csc(t) - y(t) \cot(t)) \, dt$. (corrected)

By taking the derivative with respect to $x$ we translate the problem to the question of finding the solution of the initial value problem:

$$y'(x) = 2x \csc(x) - y(x) \cot(x), \ y(1) = 1.$$

We found the general solution of this differential equation in Exam 1:

$$y = \frac{1}{\sin(x)} (x^2 + A).$$

The initial condition says

$$1 = \frac{1}{\sin(1)} (1 + A).$$
Hence the solution is
\[ y = \frac{1}{\sin(x)} \left( x^2 + \sin(1) - 1 \right). \]

2. \( y(x) = 2 + \int_{1}^{x} (-y(t)(t + 2y(t)^{-1})^{-1}) \, dt. \)

By taking the derivative with respect to \( x \) we translate the problem to the question of finding the solution of the initial value problem:

\[ (x + 2y^{-1})y' + y = 0, \quad y(1) = 2. \]

We found the general solution of this differential equation in Exam 1:

\[ xy + \ln(y^2) = C. \]

The initial condition says

\[ 2 + \ln(4) = C. \]

Thus the solution is given by

\[ xy + \ln(y^2) = 2 + \ln(4). \]

In the following exercises consider the following initial value problem:

\[ y' = \sin(x) + \cos(y), \quad y(0) = 0, \quad |x| < 10, \quad |y| < 10. \]

3. What are the minimal constants \( M \) and \( C \) of the theorem in this case?

Here, \( f(x, y) = \sin(x) + \cos(y) \). The maximum of the function \( |f(x, y)| \) for \( |x| < 10, \ |y| < 10 \) is \( \sin(\frac{\pi}{2}) + \cos(0) = 1 + 1 = 2 \). Thus \( M = 2 \). Since the derivative \( \partial_y f(x, y) = -\sin(y) \), the maximum of \( |\partial_y f(x, y)| \) for \( |x| < 10, \ |y| < 10 \) is 1. Thus \( C = 1 \).

4. Given \( M \) and \( C \) as in Problem 3, what is the upper bound on the number \( h \)?

Since, \( \alpha = \beta = 10 \), the upper bound is

\[ \min\{\alpha, \frac{\beta}{M + \beta C}\} = \min\{10, \frac{10}{2 + 10}\} = \frac{10}{12}. \]

5. Suppose \( h = \frac{3}{4} \). What is the minimal \( L \) such that

\[ |y(x) - y_L(x)| < \frac{1}{20} \quad (|x - x_0| < h)? \]
Since,
\[ Mh \frac{(Ch)^L}{1 - Ch} = 2 \cdot 3 \cdot \frac{\left(\frac{3}{4}\right)^L}{1 - \frac{3}{4}} = 6 \left(\frac{3}{4}\right)^L, \]
the condition is
\[ 6 \left(\frac{3}{4}\right)^L < \frac{1}{20}. \]
Thus
\[ L > \frac{\ln(120)}{\ln(4/3)} = 16.64. \]
Hence
\[ L = 17. \]

6. Compute \( y_1(x) \).

By definition,
\[ y_1(x) = 0 + \int_0^x (\sin(t) + \cos(0)) \, dt = \int_0^x (\sin(t) + 1) \, dt = 1 + x - \cos(x). \]

7. For what values of \( h \) does the theorem guarantee the following estimate
\[ |y(x) - y_1(x)| < \frac{1}{20} \quad (|x - x_0| < h)? \]

We need to ensure
\[ Mh \frac{Ch}{1 - Ch} < \frac{1}{20}. \]
Thus
\[ 2h \frac{h}{1 - h} < \frac{1}{20}, \]
This is a quadratic inequality for \( 0 < h < 1 \). Hence
\[ h < \frac{\sqrt{161} - 1}{80} = 0.1461. \]