Find the general solution of each of the following differential equations.

1. \( y'' + y' - 6y = 0. \)

This is a homogeneous second order differential equation with constant coefficients. The corresponding characteristic equation is

\[ p^2 + p - 6 = 0, \]

which has two solutions \( p = -3 \) and \( p = 2. \) Hence the general solution is

\[ y = Ae^{-3x} + Be^{2x}. \]

2. \( y'' + 8y = 0. \)

This is a homogeneous second order differential equation with constant coefficients. The corresponding characteristic equation is

\[ p^2 + 8 = 0, \]

which has two solutions \( p = 2\sqrt{2}i \) and \( p = -2\sqrt{2}i. \) Hence the general solution is

\[ y = Ae^{2\sqrt{2}ix} + Be^{-2\sqrt{2}ix}. \]

3. \( y'' + 2y' + 3y = 0. \)

This is a homogeneous second order differential equation with constant coefficients. The corresponding characteristic equation is

\[ p^2 + 2p + 3 = 0, \]

which has two solutions \( p = -1 - \sqrt{2}i \) and \( p = -1 + \sqrt{2}i. \) Hence the general solution is

\[ y = Ae^{(-1-\sqrt{2}i)x} + Be^{(-1+\sqrt{2}i)x}. \]

4. \( y'' + 3y' - 10y = 6e^x. \)

This is a second order differential equation with constant coefficients. The corresponding homogeneous equation is

\[ y'' + 3y' - 10y = 0. \]
The characteristic equation of the homogeneous equation is

\[ p^2 + 3p - 10 = 0, \]

which has two solutions \( p = -5 \) and \( p = 2 \). Hence the general solution of the homogeneous equation is

\[ y_h = Ae^{-5x} + Be^{2x}. \]

We look for a particular solution of the original equation, of the form \( y_p = Ce^x \). Then

\[ y_p'' + 3y_p' - 10y_p = -6Ce^x. \]

Hence,

\[ -6Ce^x = 6e^x. \]

Thus, \( C = -1, \ y_p = -e^x \) and therefore the solution of our equation is

\[ y = Ae^{-5x} + Be^{2x} - e^x. \]

5. \( y'' + 3y' - 10y = 6e^{4x} \).

This is a second order differential equation with constant coefficients. The corresponding homogeneous equation is

\[ y'' + 3y' - 10y = 0. \]

The general solution of the homogeneous equation is

\[ y_h = Ae^{-5x} + Be^{2x}. \]

We look for a particular solution of the original equation, of the form \( y_p = Ce^{4x} \). Then

\[ y_p'' + 3y_p' - 10y_p = (16C + 12C - 10C)e^x = 18Ce^{4x}. \]

Hence,

\[ 18Ce^{4x} = 6e^{4x}. \]

Thus, \( C = \frac{1}{3}, \ y_p = \frac{1}{3}e^{4x} \) and therefore the solution of our equation is

\[ y = Ae^{-5x} + Be^{2x} + \frac{1}{3}e^{4x}. \]

6. \( y'' + y = 2\cos(x) \).

The solution of the homogeneous equation

\[ y'' + y = 0 \]
is

\[ y_h = Ae^{ix} + Be^{-ix}. \]

Since

\[ \cos''(x) + \cos(x) = 0, \]

we look for the particular solution of the form

\[ y_p = x(\cos(x) + \sin(x)). \]

Then

\[ y_p' = \cos(x) + \sin(x) + x(-\sin(x) + \cos(x)), \]
\[ y_p'' = 2(-\sin(x) + \cos(x)) + x(-\cos(x) - \sin(x)), \]
\[ y_p'' + y_p = 2(-\sin(x) + \cos(x)). \]

Thus

\[ 2(-\sin(x) + \cos(x)) = 2\cos(x). \]

Hence, \( a = 0 \) and \( b = 1 \). Therefore \( y_p = x\sin(x) \). The general solution of the original equation is

\[ y = Ae^{ix} + Be^{-ix} + x\sin(x). \]

7. \( y'' + y' = 10x^4 + 2. \)

The corresponding homogeneous equation is

\[ y'' + y' = 0. \]

The characteristic equation

\[ p^2 + p = 0 \]

has two solutions \( p = 0 \) and \( p = -1 \). Hence the solution of the homogeneous equation is

\[ y_h = A + Be^{-x}. \]

We look for a particular solution of the form \( y_p = x(Ax^4 + Bx^3 + Cx^2 + Dx + E) = Ax^5 + Bx^4 + Cx^3 + Dx^2 + Ex \). Then

\[ y_p' = 5Ax^4 + 4Bx^3 + 3Cx^2 + 2Dx + E, \]
\[ y_p'' = 20Ax^3 + 12Bx^2 + 6Cx + 2D, \]
\[ y_p'' + y_p' = 5Ax^4 + (20A + 4B)x^3 + (12B + 3C)x^2 + (6C + 2D)x + 2D + E. \]

Hence,

\[ 10 = 5A, \]
\[ 0 = 20A + 4B, \]
\[ 0 = 12B + 3C, \]
\[ 0 = 6C + 2D, \]
\[ 2 = 2D + E. \]
Thus
\[ y_p = 2x^5 - 10x^4 + 40x^3 - 120x^2 + 242x, \]
so that
\[ y = A + Be^{-x} + 2x^5 - 10x^4 + 40x^3 - 120x^2 + 242x. \]

8. \( y'' - 2y' + y = 6e^x. \)

The corresponding homogeneous equation is
\[ y'' - 2y' + y = 0. \]

The characteristic equation
\[ p^2 - 2p + 1 = 0 \]
has one solution \( p = 1. \) Hence the solution of the homogeneous equation is
\[ y_h = Ae^x + Bxe^x. \]

We look for a particular solution of the form \( y_p = Cx^2e^x. \) Then
\[ y'_p = 2Cxe^x + Cx^2e^x, \]
\[ y''_p = 2Ce^x + 4Cxe^x + Cx^2e^x, \]
\[ y''_p - 2y'_p + y_p = 2Ce^x. \]

Hence,
\[ C = 3. \]

Thus
\[ y_p = 3x^2e^x, \]
so that
\[ y = Ae^x + Bxe^x + 3x^2e^x. \]