ODE sec. 3, Spring 2007

Homework 1

1. State the Fundamental Theorem of Calculus (i.e. copy it from a book).

Let \( f \) be a continuous function defined on a closed interval \([a, b]\), with values in the real numbers. Then the formula

\[
\int_a^x f(t) \, dt \quad (a < x < b)
\]

defines a differentiable function of \( x \) and

\[
\frac{d}{dx} \int_a^x f(t) \, dt = f(x) \quad (a < x < b).
\]

2. State the following basic properties of the exponential function:
   (a) \( e^{a+b} = e^a e^b \),
   (b) \( \frac{d}{dx} e^x = e^x \).

3. State the following basic properties of the natural logarithm:
   (a) \( \ln(ab) = \ln(a) + \ln(b) \),
   (b) \( \frac{d}{dx} \ln(x) = x^{-1} \),
   (c) \( \int \frac{1}{x} \, dx = \ln(|x|) \).

4. State the Chain Rule (i.e. copy it from a book).

\[
\frac{d}{dx} f(g(x)) = f'(g(x))g'(x).
\]

5. Determine if the following equation is exact and solve:

\[
(x + 2y^{-1})dy + ydx = 0.
\]

Here \( M(x, y) = y \), \( N(x, y) = x + 2y^{-1} \) and \( \partial_y M = 1 = \partial_x N \). Thus the equation is exact.

We look for a function \( f(x, y) \) such that

\[
\partial_x f = M = y.
\]

Thus

\[
f(x, y) = xy + \phi(y).
\]

On the other hand

\[
x + 2y^{-1} = \partial_y f = x + \phi'(y).
\]
Therefore \[ \phi'(y) = 2y^{-1}, \]
so that \[ \phi(y) = \ln(y^2). \]
Hence, \( f(x, y) = xy + \ln(y^2) \) and the solution is given by the equation \[ xy + \ln(y^2) = C, \]
where \( C \) is an arbitrary constant.

6. Determine if the following equation is exact and solve:

\[
\frac{xdx}{(x^2 + y^2)^{3/2}} + \frac{ydy}{(x^2 + y^2)^{3/2}} = 0.
\]

Here \( M(x, y) = \frac{x}{(x^2 + y^2)^{3/2}} \), \( N(x, y) = \frac{y}{(x^2 + y^2)^{3/2}} \).

\[
\partial_y M = x(-3/2)(x^2 + y^2)^{-5/2}2y \quad \text{and} \quad \partial_x N = y(-3/2)(x^2 + y^2)^{-5/2}2x.
\]

Thus \( \partial_y M = \partial_x N \) so that the equation is exact.

We look for a function \( f(x, y) \) such that

\[
\partial_x f = M = \frac{x}{(x^2 + y^2)^{3/2}}.
\]

Thus

\[
f(x, y) = -(x^2 + y^2)^{-1/2} + \phi(y).
\]

On the other hand

\[
\frac{y}{(x^2 + y^2)^{3/2}} = \partial_y f = \frac{y}{(x^2 + y^2)^{3/2}} + \phi'(y).
\]

Therefore

\[
\phi'(y) = 0,
\]

so that

\[
\phi(y) = A,
\]

and we may assume that \( A = 0 \). Hence, \( f(x, y) = -(x^2 + y^2)^{-1/2} \) and the solution is given by the equation \( -(x^2 + y^2)^{-1/2} = C \),
where \( C \) is an arbitrary negative constant. The last equation may be rewritten as

\[
x^2 + y^2 = \left( \frac{1}{-C} \right)^2.
\]

Thus the solution curves are circles centered at the origin.
7. Sketch the following family of curves, find the orthogonal family add those to your sketch:

\[ y = c(1 + \cos(x)). \]

This family satisfies the following differential equation

\[ y' = -\cos(x) = -\frac{y}{1 + \cos(x)} \sin(x). \]

Replacing \( y' \) by \(-\frac{1}{y'}\) we obtain the following equation

\[ -\frac{1}{y'} = -\frac{y}{1 + \cos(x)} \sin(x), \]

or equivalently

\[ y' = \frac{1 + \cos(x)}{y \sin(x)}. \]

This is a separable equation

\[ y \, dy = \frac{1 + \cos(x)}{\sin(x)} \, dx. \]

We integrate and get

\[ \frac{1}{2} y^2 = \int \csc(x) \, dx + \int \cot(x) \, dx = \ln(|\csc(x) - \cot(x)|) + \ln(|\sin(x)|) + A. \]

Hence,

\[ y^2 = \ln((\csc(x) - \cot(x))^2) + \ln(\sin^2(x)) + B, \]

where \( B = 2A \) is an arbitrary constant. This is the equation describing the orthogonal family of curves.

8. Sketch the following family of curves, find the orthogonal family add those to your sketch:

\[ x + y = c. \]

This family satisfies the following differential equation

\[ 1 + y' = 0. \]

Replacing \( y' \) by \(-\frac{1}{y'}\) we obtain the following equation

\[ 1 - \frac{1}{y'} = 0. \]
Equivalently,  
\[ y' = 1. \]

The solution of this last equation is  
\[ y = x + C. \]

This is the equation describing the orthogonal family of curves (in this case straight lines).

9. Verify that the following equation is homogeneous and solve it:  
\[ (x^2 - 2y^2)dx + xydy = 0. \]

We solve for \( y' = \frac{dy}{dx} \):  
\[ y' = \frac{2y^2 - x^2}{xy} = 2\frac{y}{x} - \frac{x}{y}. \]

We see that this is a homogeneous equation and use the substitution \( z = \frac{y}{x} \) to solve it.  
\[ y' = xz' + z, \]
\[ xz' + z = 2z - z^{-1}, \]
\[ xz' = z - z^{-1} = \frac{z^2 - 1}{z}. \]

The last equation is separable:  
\[ \frac{z}{z^2 - 1} \, dz = \frac{dx}{x}. \]

Hence, via an integration  
\[ \ln(|z^2 - 1|) = \ln(x^2) + A. \]

Therefore  
\[ z^2 - 1 = Bx^2. \]

Hence  
\[ y^2 = Bx^4 + x^2. \]

10. Solve the following differential equation:  
\[ yy'' - (y')^2 = 0. \]
Here we reduce the order of the equation via the substitution $z = y'$, so that $y'' = z \frac{dz}{dy}$. This leads to the following equation

$$yz \frac{dz}{dy} = z^2.$$ 

This is a separable equation:

$$\frac{dz}{z} = \frac{dy}{y}.$$ 

Hence,

$$z = Ay.$$ 

But $z = y'$. We integrate and get

$$y = Be^{Ax},$$

where $A$ and $B$ are arbitrary constants.