1. Verify that the following functions are solutions of the corresponding differential equations:

(a) $y = ce^{kx}, \quad y' = ky,$

By the chain rule, the derivative of $y$ is equal to

$$y' = c ke^{kx} = k \cdot ce^{kx} = ky.$$  

(b) $y = c_1 \cos(3x) + c_2 \sin(3x), \quad y'' + 9y = 0,$

By the chain rule, the second derivative of $\cos(3x)$ is equal to $-3^2 \cos(3x)$, and similarly for $\sin(3x)$. Hence,

$$y'' + 9y = c_1(-3^2 \cos(3x)) + c_2(-3^2 \sin(3x)) + 9y = -9(c_1 \cos(3x) + c_2 \sin(3x)) + 9y = 0.$$  

(c) $y = c_1 \cosh(3x) + c_2 \sinh(3x), \quad y'' - 9y = 0.$

By the chain rule, the second derivative of $\cosh(3x)$ is equal to $3^2 \cosh(3x)$, and similarly for $\sinh(3x)$. Hence,

$$y'' - 9y = c_1(3^2 \cos(3x)) + c_2(3^2 \sin(3x)) - 9y = 9(c_1 \cos(3x) + c_2 \sin(3x)) - 9y = 0.$$  

2. For each of the following differential equations find the particular solution which satisfies the given initial condition:

(a) $y' = xe^x, \quad y = 3$ when $x = 1$,

This is a separable equation:

$$dy = xe^x \, dx.$$  

We integrate both sides (the right hand side by parts) and get

$$y = \int xe^x \, dx = xe^x - \int e^x \, dx = xe^x - e^x + C.$$  

Since, $y = 3$ when $x = 1$, we see that

$$3 = e^1 - e^1 + C,$$

i.e. $C = 3$. Thus the solution is $y = xe^x - e^x + 3$.

(b) $y' = \ln(x), \quad y = 0$ when $x = e$,

This is a separable equation:

$$dy = \ln(x) \, dx.$$
We integrate both sides (the right hand side by parts) and get

\[ y = \int \ln(x) \, dx = x\ln(x) - \int x \cdot \frac{1}{x} \, dx = x\ln(x) - x + C. \]

Since, \( y = 0 \) when \( x = e \) and \( \ln(e) = 1 \), we see that

\[ 0 = e \ln(e) - e + C = C, \]

i.e. \( C = 0 \). Thus the solution is \( y = x\ln(x) - x \).

3. Show that the function

\[ y(x) = e^{x^2} \int_0^x e^{-t^2} \, dt \]

is a solution of the differential equation \( \frac{dy}{dx} = 2xy + 1 \).

By the Fundamental Theorem of Calculus,

\[ \frac{d}{dx} \int_0^x e^{-t^2} \, dt = e^{-x^2}. \]

Hence, by the product rule,

\[ y' = \left( \frac{d}{dx} e^{x^2} \right) \int_0^x e^{-t^2} \, dt + e^{x^2} \left( \frac{d}{dx} \int_0^x e^{-t^2} \, dt \right) = 2xe^{x^2} \int_0^x e^{-t^2} \, dt + e^{x^2} e^{-x^2} = 2x \left( e^{x^2} \int_0^x e^{-t^2} \, dt \right) + 1 = 2xy + 1. \]

4. Draw the field of slopes for the differential equation:

(a) \( y' = y \),

(b) \( y' = x + y \),

4. Use the method of separation of variables to the general solution of the following differential equation:

(a) \( y' + y\tan(x) = 0 \),

This is a separable equation:

\[ \frac{dy}{y} = -\frac{\sin(x)}{\cos(x)} \, dx. \]

Since the derivative of \( \cos \) is \( -\sin \), the integration of two sides of the above equation gives

\[ \ln(|y|) = \ln(|\cos(x)|) + A, \]
where \( A \) is an arbitrary constant. We apply the exponential function to both sides and get

\[ |y| = e^A |\cos(x)|. \]

Hence

\[ y = \pm e^A \cos(x). \]

Let \( B = \pm e^A \). Then \( B \) is an arbitrary constant and

\[ y = B \cos(x). \]

(b) \( y \ln(y) - xy' = 0 \).

This is a separable equation:

\[ \frac{dx}{x} = \frac{dy}{y \ln(y)}, \]

and since the \( y \) is under the logarithm, we must have \( y > 0 \). The integration of two sides of the above equation gives

\[ \ln(|x|) + A = \ln(|\ln(y)|), \]

where \( A \) is an arbitrary constant. We apply the exponential function to both sides and get

\[ e^A |x| = |\ln(y)|. \]

Thus,

\[ |\ln(y)| = \pm e^A |x|. \]

Let \( B = \pm e^A \). Then \( B \) is an arbitrary constant and

\[ \ln(y) = Bx. \]

We apply the exponential function again and get

\[ y = e^{Bx}. \]

5. Find the general solution of the following first order linear differential equation:
   (a) \( y' + xy = x \),

Since \( \int x \, dx = \frac{1}{2} x^2 \), we multiply both sides by \( e^{\frac{1}{2}x^2} \):

\[ e^{\frac{1}{2}x^2} y' + x e^{\frac{1}{2}x^2} y = x e^{\frac{1}{2}x^2}. \]

The left hand side is equal to

\[ \frac{d}{dx} \left( e^{\frac{1}{2}x^2} y \right). \]
Therefore,
\[
\frac{d}{dx}(e^{\frac{1}{2}x^2} y) = xe^{\frac{1}{2}x^2}.
\]
Thus
\[
e^{\frac{1}{2}x^2} y = \int xe^{\frac{1}{2}x^2} dx = e^{\frac{1}{2}x^2} + A,
\]
where \(A\) is an arbitrary constant. Finally we solve for \(y\):
\[
y = e^{-\frac{1}{2}x^2} \left( e^{\frac{1}{2}x^2} + A \right) = 1 + A e^{-\frac{1}{2}x^2}.
\]

(b) \(y' + ycot(x) = 2xcsc(x)\).

Since the integral of \(cot\) is equal to \(ln(|sin|)\), we multiply both sides by \(sin(x)\) and get
\[
sin(x) (y' + ycot(x)) = sin(x)2xcsc(x),
\]
or equivalently
\[
sin(x)y' + cos(x)y = 2x.
\]
The left hand side is equal to
\[
\frac{d}{dx} (sin(x)y).
\]
Therefore,
\[
\frac{d}{dx} (sin(x)y) = 2x.
\]
Thus
\[
sin(x)y = \int 2x dx = x^2 + A,
\]
where \(A\) is an arbitrary constant. Finally we solve for \(y\):
\[
y = \frac{1}{sin(x)} (x^2 + A) = csc(x) \left( x^2 + A \right).
\]

6. A tank contains 40 gal of pure water. Brine with 3lb of salt per gallon flows in at the rate of 2 gal/min. The thoroughly stirred mixture then flows out at the rate of 3 gal/min.

(a) Find the amount of salt in the tank at time \(t\).
(b) Find the amount of salt in the tank when the brine in it has been reduced to 20 gal.
(c) When is the amount of salt in the tank greatest?
Let $x(t)$ denote the amount of salt in the tank at time $t$, and let $v(t)$ denote the amount of the brine in the tank at time $t$. Then $v(t) = 40 - t$, $0 \leq t \leq 40$ and

$$x'(t) = 6 - 3 \frac{x(t)}{v(t)}.$$  

Thus

$$x'(t) - \frac{3}{t - 40} x(t) = 6.$$  

This is a first order linear equation. Since

$$\int -\frac{3}{t - 40} \, dt = -3 \ln|t - 40| = -3 \ln(40 - t) = \ln((40 - t)^{-3}),$$

we multiply both sides of the equation (6.2) by $(40 - t)^{-3}$ and obtain

$$(40 - t)^{-3}(x'(t) - \frac{3}{t - 40} x(t)) = 6(40 - t)^{-3}.$$  

The left hand side of the equation (6.3) is equal to

$$\frac{d}{dt} ((40 - t)^{-3} x(t)).$$

Hence,

$$(40 - t)^{-3} x(t) = \int 6(40 - t)^{-3} \, dt = 3(40 - t)^{-2} + A,$$

where $A$ is a constant. Therefore,

$$x(t) = 3(40 - t) + A(40 - t)^3.$$  

Since at the beginning the tank contains pure water, $x(0) = 0$. Hence,

$$0 = 3 \times 40 + A40^3,$$

which implies

$$A = -3 \times 40^{-2} = -\frac{3}{1600}.$$  

Thus

$$x(t) = 3(40 - t) - \frac{3}{1600}(40 - t)^3.$$  

This completes part (a). For (b), we see that the time when “the brine in it has been reduced to 20 gal” may be found from the equation $40 - t = 20$. Thus $t = 20$. At that time

$$x(20) = 3(40 - 20) - \frac{3}{1600}(40 - 20)^3 = 60 - \frac{3}{1600}8000 = 45.$$  

This solves (b). Finally, we need to find $t$ such that $x(t)$ is maximal. In particular $x'(t) = 0$. Thus, by (6.4),

$$-3 + \frac{9}{1600}(40 - t)^2 = 0,$$

i.e.

$$t = 40 - \left(\frac{1600}{3}\right)^{1/2} = 16.9.$$  

Since $x(t) \geq 0$, $x(0) = x(40) = 0$, the unique critical point $t = 16.9$ is the place where the function has the maximum. This is the solution for part (c).