1. Find the area of the parallelogram spanned by the vectors \( u = (1, 0, 1) \) and \( v = (1, 1, 3) \).

We use the fact that the area of the parallelogram spanned by the vectors \( u \) and \( v \) is equal to the norm of the cross product \( u \times v \). The cross product

\[
u \times v = (i + k) \times (i + j + 3k) = i \times j + 3i \times k + k \times i + k \times j = -i - 2j + k,
\]

and the norm

\[\| u \times v \| = \sqrt{(-1)^2 + (-2)^2 + 1^2} = \sqrt{6}.\]

This is the area in question.

2. Find an equation (in terms of \( x, y \) and \( z \)) of the plane passing through the points \((1, 0, 1), (1, 1, 3)\) and \((3, 4, 5)\).

This the plane passing through the point \((1, 0, 1)\) and parallel to the vectors \( u = (1, 1, 3) - (1, 0, 1) = (0, 1, 2) = j + 2k \) and \( v = (3, 4, 5) - (1, 0, 1) = (2, 4, 4) = 2i + 4j + 4k \). Thus it is the same as the plane passing through the point \((1, 0, 1)\) and orthogonal to \( u \times v \). As in the previous problem, we compute

\[
u \times v = -4i + 4j - 2k.
\]

Thus the equation of the plane is

\[-4(x - 1) + 4(y - 0) - 2(z - 1) = 0,
\]
or equivalently,

\[2x - 2y + z = 3.
\]

3. Find a parametric description of the line tangent to the curve \( r(t) = (t + 1, e^{2t}, e^t) \) at the point \( r_0 = (1, 1, 1) \).

Notice first that

\[r(t) = (t + 1, e^{2t}, e^t) = (1, 1, 1)
\]

if and only if \( t = 0 \). Thus we know precisely ”the time” when the point \( r(t) \) on our curve coincides with \( r_0 \). We compute the ”velocity” at \( t = 0 \):

\[r'(0) = (1, 2e^{2t}, e^t)|_{t=0} = (1, 2, 1).
\]

Thus the parametric description of the tangent line is

\[(x(t), y(t), z(t)) = (1, 1, 1) + t(1, 2, 1), \quad (t \in \mathbb{R}),\]
or equivalently,

\[ x(t) = 1 + t, \]
\[ y(t) = 1 + 2t, \]
\[ z(t) = 1 + t, \]

where \( t \in \mathbb{R} \).

4. Find an equation of the plane tangent to the surface \( z = x^2 + y^2 \) at the point \((1, 1, 2)\).

We use the formula

\[ z - z_0 = \partial_x f(x_0, y_0)(x - x_0) + \partial_y f(x_0, y_0)(y - y_0). \]

Here, \( z_0 = 2 \), \( f(x, y) = x^2 + y^2 \), \( x_0 = 1 \) and \( y_0 = 1 \). Hence,

\[ \partial_x f(x_0, y_0)|_{x_0=1, y_0=1} = 2x_0|_{x_0=1} = 2, \text{ and } \partial_y f(x_0, y_0)|_{x_0=1, y_0=1} = 2y_0|_{y_0=1} = 2, \]

and consequently the equation of the tangent plane is

\[ z - 2 = 2(x - 1) + 2(y - 1), \]

or equivalently,

\[ z = 4x + 4y - 2. \]

5. Give an example of a function \( f(x, y) \) and a region \( S \subseteq \mathbb{R}^2 \) such that

\[ \int \int_S f(x, y) \, dx \, dy = 20. \]

Let \( S = \{(x, y); 0 \leq x \leq 1, 0 \leq y \leq 1\} \). This is the square with vertices \((0, 0)\), \((1, 0)\), \((1, 1)\) and \((0, 1)\). In particular the area of \( S \) is 1.

Let \( f(x, y) = 20 \) for all \( x \) and \( y \). Then the integral in question is equal to 20 times the area of \( S \), which is 20.

6. Let \( S = \{(x, y); x^2 + y^2 \leq 1, y \geq 0\} \). This is the part of the disc of radius 1 which is in the upper half plane. Compute the integral

\[ \int \int_S y \, dx \, dy. \]

We compute in polar coordinates:

\[
\int \int_S y \, dx \, dy \\
= \int_0^\pi \int_0^1 r \sin(\theta) \, r \, dr \, d\theta \\
= \int_0^1 r^2 \, dr \int_0^\pi \sin(\theta) \, d\theta \\
= \frac{1}{3} \cos(\theta)|_0^\pi = \frac{1}{3} \cdot 2 = \frac{2}{3}.
\]
7. Sketch the vector field \( F(x, y) = \left( \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right) \).

This is the radial vector field which looks like the one right under Exercise 31 on page 1097 in the book, where each arrow has length 1.

8. Parametrize the line segment which begins at \((1, 0)\) and ends at \((1, 7)\).

\[
x(t) = 1, \quad y(t) = t, \quad 0 \leq t \leq 7.
\]

9. Let \( C \) be the line segment of Problem 9. Let \( F(x, y) = (y, -x) \). Compute the integral.

\[
\int \int_{C} F \cdot n \, ds.
\]

We use the formula

\[
\int \int_{C} F \cdot n \, ds = \int_{t_0}^{t_1} (F_1(x(t), y(t))y'(t) - F_2(x(t), y(t))x'(t)) \, dt.
\]

Here, \( F_1(x(t), y(t)) = y(t) = t, \quad y'(t) = 1, \quad x'(t) = 0 \) and \( 0 \leq t \leq 7 \). Hence,

\[
\int \int_{C} F \cdot n \, ds = \int_{0}^{7} t \, dt = \frac{49}{2}.
\]

10. Give an example of a vector field \( F(x, y) \) and a curve \( C \) such that such that

\[
\int \int_{C} F \cdot n \, ds = 10.
\]

If we multiply the vector field \( F \) of Problem 9 by \( 10 \frac{2}{49} \), the integral gets multiplied by the same number. Hence, \( F(x, y) = (\frac{20}{49}y, -\frac{20}{49}x) \) and the same curve as in Problem 9 will do.