ON THE MOMENT MAP OF A MULTIPLICITY FREE ACTION

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The purpose of this note is to show that the Orbit Conjecture of C. Benson, J. Jenkins, R. L. Lipsman and G. Ratcliff [BJLR1] is true. Another proof of that fact has been given by those authors in [BJLR2]. Their proof is based on their earlier results, announced together with the conjecture in [BJLR1]. We follow another path: using a geometric quantization result of Guillemin–Sternberg [G-S] we reduce the conjecture to a similar statement for a projective space, which is a special case of a characterization of projective smooth spherical varieties due to Brion [B2].

Let \( V \) be a finite-dimensional complex representation space for a connected reductive complex group \( G \). Choose a maximal compact subgroup \( K \subseteq G \) and a \( K \)-invariant positive definite hermitian form \((\cdot, \cdot)\) on \( V \). Let

\[
\quad (u, v) = \text{Im} (u, v) \quad (u, v \in V)
\]

be the associated symplectic form. Recall the unnormalized moment map

\[
\quad \tau_t : V \to \mathfrak{t}^*, \quad \tau_t(v)(X) = \langle X(v), v \rangle \quad (v \in V),
\]

and the normalized moment map

\[
\quad \mu_t : P(V) \to \mathfrak{t}^*, \quad \mu_t(\tilde{v})(X) = \frac{\langle X(v), v \rangle}{(v, v)} \quad (v \in V),
\]

where \( P(V) \) is the projective space of lines in \( V \) and \( \tilde{v} \) is the line passing through \( v \). It is easy to see that these maps are \( K \)-equivariant.

Let \( \mathbb{C}[V] \) be the space of polynomial functions on \( V \). Clearly the group \( K \) acts on \( \mathbb{C}[V] \). Recall from [BJLR1] that the action of \( K \) on \( V \) is called multiplicity-free if the action of \( K \) on \( \mathbb{C}[V] \) has no multiplicities, i.e. the multiplicities of the irreducible representations of \( K \) in \( \mathbb{C}[V] \) are at most one.

Here is the Orbit Conjecture (see [BJLR1]), stated as a theorem.

1991 Mathematics Subject Classification: Primary 22E45; Secondary 14L30.
Theorem. The map $\tau_\mathfrak{k}$ is one-to-one on $K$-orbits (i.e. distinct orbits are mapped onto distinct orbits) if and only if the action of $K$ on $V$ is multiplicity-free.

Before we give the proof of the theorem, we will recall a result of Brion on moment maps of smooth projective $G$-varieties.

An algebraic variety $X$ with an action of a complex reductive group $G$ is called spherical if some (or equivalently each) Borel subgroup $B$ of $G$ has a dense orbit in $X$. It is well known (see [Se]) that an affine $G$-variety $X$ is spherical if and only if it is multiplicity-free, i.e. its ring $\mathbb{C}[X]$ of polynomial functions has no multiplicities as a $G$-module. For a good introduction to the theory of spherical varieties the reader may consult [B1].

Assume that the variety $X$ is contained in the projective space $\mathbb{P}(V)$ for some complex representation space $V$ of $G$, and that the action of $G$ on $X$ is induced by that on $V$. Let $\mu_X : X \to \mathfrak{k}^*$ be the normalized moment map of $X$, i.e. the composite $X \hookrightarrow \mathbb{P}(V) \to \mathfrak{k}^*$ of the normalized moment map (3) and inclusion. Assume that $X$ is smooth and projective (closed in $\mathbb{P}(V)$). Then the theorem of Brion (see [B2, 5.1], [B1, Theorem 3.2]) says that

(4) $X$ is spherical if and only if $\mu_X$ is one-to-one on $K$-orbits.

Proof of the theorem. We notice first that

(5) if $\tau_\mathfrak{k}$ is one-to-one on $K$-orbits, then so is the normalized moment map $\mu_\mathfrak{k}$.

Indeed, we can view this normalized map as the restriction of $\tau_\mathfrak{k}$ to the unit sphere $S$ in $V$ composed with the canonical map $S \to \mathbb{P}(V)$.

Let $U$ be the full isometry group of the hermitian form $(\cdot, \cdot)$. We have $K \subseteq U$. Let $Z$ denote the center of $U$. Let $\mathcal{P}_d(V) \subseteq \mathbb{C}[V]$ be the subspace of homogeneous polynomials of degree $d$. Then the spaces $\mathcal{P}_d(V)$ are the eigenspaces for the action of $Z$ on $\mathbb{C}[V]$, corresponding to distinct eigenvalues (weights). Notice that

(6) if $Z \subseteq K$ and if the map $\mu_\mathfrak{k}$ is one-to-one on $K$-orbits, then so is the unnormalized map $\tau_\mathfrak{k}$.

Indeed, the restriction of $\tau_\mathfrak{k}$ to any sphere in $V$ is one-to-one on $K$-orbits and the composition of $\tau_\mathfrak{k}$ with the restriction map $\mathfrak{k}^* \to \mathfrak{z}^*$ distinguishes the spheres.

Clearly

(7) if $Z \subseteq K$, then $\mathbb{P}(V)$ is spherical if and only if $V$ is spherical.

This is obvious because under the assumption (7), $\mathbb{C}^\times$-identity is contained in every Borel subgroup of $G$.

By combining (4), for $X = \mathbb{P}(V)$, with (5)--(7) we see that the theorem holds if $Z \subseteq K$. 
Assume from now on that $Z$ is not contained in $K$.

Suppose $\tau_k$ is one-to-one on $K$-orbits. Then by (4) and (5), $P(V)$ is $G$-spherical. Hence $V$ is $(\mathbb{C}^* \cdot G)$-spherical. Hence the group $Z \cdot K$ acts on $\mathbb{C}[V]$ without multiplicities. Therefore $K$ acts on each $P_d(V)$ without multiplicities.

Recall that each $P_d(V)$ is irreducible for the action of $U$. Let $O_d \subseteq u^*$ denote the corresponding orbit, as in [G-S, Theorem 3.7]. This is the coadjoint orbit passing through a highest weight of this representation, divided by $2\pi i$. Then it is easy to see that $O_d \subseteq \tau_u(V)$, where $\tau_u : V \to u^*$ is defined as in (3). This map is one-to-one on $U$-orbits. In fact, $V_d = \tau_u^{-1}(O_d)$ is a sphere of radius $d \cdot \text{const}$, where the const does not depend on $d$.

Let $q : u^* \to \mathfrak{t}^*$ be the restriction map. Then $\tau_k = q \circ \tau_u$. Suppose $\pi \in \hat{K}$ occurs in $\mathbb{C}[V]$ at least twice. Then it occurs in $P_d(V)$ and in $P_d'(V)$ for some $d \neq d'$. Let $O_\pi \subseteq \mathfrak{t}^*$ be the corresponding orbit (as in [G-S]). Then by [G-S, Theorem 6.3],

\begin{equation}
O_\pi \subseteq q(O_d) = \tau(V_d) \quad \text{and} \quad O_\pi \subseteq q(O_{d'}) = \tau(V_{d'}).\end{equation}

But $V_d$ and $V_{d'}$ are spheres of distinct radii. Hence (9) contradicts the assumption that $\tau_k$ was one-to-one on $K$-orbits.

Conversely, suppose $K$ acts on $\mathbb{C}[V]$ without multiplicities. Then $P(V)$ is spherical. Hence $\mu_k$ is one-to-one on $K$-orbits. Therefore the map $V/(Z \cdot K) \to \mathfrak{t}^*/K$ induced by $\tau_k$ is one-to-one. Thus it will suffice to show that each $(Z \cdot K)$-orbit in $V$ is a $K$-orbit.

It is well known (see [O-V, p. 138]) that functions in the algebra $\mathbb{C}[V_K] K$ separate $K$-orbits. As a $U$-module, $\mathbb{C}[V_K] = \mathbb{C}[V] \otimes \mathbb{C}[V]^c$, where the superscript $c$ indicates the contragredient. Let $\mathbb{C}[V] = \sum \pi$ be the decomposition into irreducible $K$-modules. Then, by Schur’s lemma, $\mathbb{C}[V_K]^K = \sum (\pi \otimes \pi^c)^K$. Hence $\mathbb{C}[V_K]^K$ consists of $Z$-invariant functions, and we are done. ■

Acknowledgments. The authors are grateful to Romuald Dąbrowski for pointing out some errors in an earlier version of this note.

REFERENCES


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Received 30 October 1995