1. Introduction. Let $G, G' \subseteq Sp(W)$ be a reductive dual pair of type I; see [H2]. Thus, there is a division algebra $D = (\mathbb{R}, \mathbb{C}, H)$ with an involution over $\mathbb{R}$, two finite-dimensional vector spaces over $D$, $V$ and $V'$ equipped with non-degenerate forms $(\cdot, \cdot)$ and $(\cdot, \cdot)'$, respectively—one hermitian and the other skew-hermitian. The groups $G, G'$ are the isometry groups of the forms $(\cdot, \cdot), (\cdot, \cdot)'$, respectively. Let $W$ denote the vector space $W = \text{Hom}(V', V)$. A symplectic form on $W$ is defined by

$$\langle w, w' \rangle = \text{tr}_{D/\mathbb{R}}(ww'^*) \quad (w, w' \in W),$$

where the map $\text{Hom}(V', V) \ni w \mapsto w^* \in \text{Hom}(V, V')$ is defined by

$$\langle w(v'), v \rangle = \langle v', w^*(v) \rangle' \quad (w \in W, v \in V, v' \in V').$$

The groups $G$ and $G'$ act on $W$ via postmultiplication and premultiplication by the inverse, respectively. These actions embed $G$ and $G'$ into the symplectic group $Sp(W)$.

Let $\tilde{Sp}$ denote the metaplectic group, and let $\tilde{G}, \tilde{G}'$ be the preimages of $G, G'$ under the covering map $\tilde{Sp} \rightarrow Sp$. The duality theorem of Howe [H3] states that there is a bijection $\Pi \leftrightarrow \Pi'$ between certain irreducible admissible representations of $\tilde{G}$ and $\tilde{G}'$.

Recall the unnormalized moment maps

$$\tau_g : W \ni w \mapsto ww^* \in g, \quad \tau_{g'} : W \ni w \mapsto w^*w \in g'.$$

In the early 1980s, Howe conjectured that the wave-front sets of $\Pi$ and $\Pi'$ are related to the geometry of moment maps in some nice way.

**Conjecture (Howe).** For a generic pair $(\Pi, \Pi')$ occurring in Howe's correspondence,

$$WF(\Pi') = \tau_{g'}(\tau_g^{-1}(WF(\Pi))).$$

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The equality (1.4) was proven in [P. 7.10] under a very strong assumption that the pair $G$, $G'$ is in the stable range, with $G$ the smaller member, and that the representation $\Pi$ is unitary and finite-dimensional.

In this paper, we propose another approach towards a proof of the conjecture (1.4). Let $\Theta_\Pi$ denote the (distribution) character of $\Pi$, and let $u_\Pi$ be the lowest term in the asymptotic expansion of $\Theta_\Pi$ at the identity, as defined in [BV]. Similarly, we have $\Theta_{\Pi'}$ and $u_{\Pi'}$. We would like to state a conjecture relating $u_n$ and $u_{n'}$ to the geometry of moment maps. Before we do it, we need some preparation.

In [H1], Howe has deduced from Witt's theorem the following.

**Theorem 1.5.** There is an open dense $G \cdot G'$-invariant subset $W_{\text{max}} \subseteq W$ such that for every orbit $\mathcal{O} \subseteq \tau_g(W_{\text{max}})$, the set $\mathcal{O}' = \tau_g(\tau_g(\mathcal{O}) \cap \mathcal{O})$ is a single $G'$ orbit.

The set $W_{\text{max}}$ is not unique, of course. In this paper, we assume that the pair $G$, $G'$ is in the stable range, with $G$ the smaller member. This means that $V'$ has an isotropic subspace of dimension greater than or equal to the dimension of $V$. We shall prove the following.

**Theorem 1.6.** There is an affine section $\sigma_g : g \to W$ to the map $\tau_g$, a function $m(z, g')$ on $g \times G'$ and a (singular) measure $\mu$ on $G'$ such that

\[ \int_W \phi(w) \, dw = \int_g \int_{G'} \int_g \phi(g \cdot \sigma_g(z) \cdot g'^{-1}) \, dg \, m(z, g') \, d\mu(g') \, dz \quad (\phi \in C_c(W)), \]

and if $\psi : g' \to \mathbb{C}$ is a continuous and rapidly decreasing function, then so is

\[ \mathcal{A} \psi(z) = \int_{G'} \psi \circ \tau_g(\sigma_g(z) \cdot g'^{-1}) m(z, g') \, d\mu(g') \quad (z \in g). \]

*Theorem 1.5 holds for the set $W_{\text{max}} = \{ g \cdot \sigma_g(z) \cdot g' \mid g \in G, z \in g, g' \in G' \}$. Furthermore, if $\psi$ is smooth and $\text{supp} \, \psi \cap \tau_g(W_{\text{max}})$ is compact, then $\mathcal{A} \psi \in S(g)$, the Schwartz space of $g$.*

For an explicit formulation, see (2.24), (2.25), (3.15), and (3.16). The case $G = Sp_{p,q}, G' = O_{-2n}$ can be treated similarly, and it is left to the reader.

Let $\mu_0$ be the canonical invariant measure on a $G$-orbit $\mathcal{O} \subseteq \mathfrak{g}$. Then by [RR], $\mu_0$ can be integrated against any rapidly decreasing function on $\mathfrak{g}$. Thus, in view of the above, we may define a measure $\mathcal{A} \mu_0$ on $g'$ by

\[ \mathcal{A} \mu_0(\psi) = \mu_0(\mathcal{A} \psi), \]

where $\psi$ is a rapidly decreasing function on $g'$. It will be clear from Theorem 1.6 and from the following construction that the measure $\mathcal{A} \mu_0$ is invariant and is supported on the closure of $\mathcal{O}'$. 
THEOREM 1.7. With the above notation, we have $\mathcal{M}_0 = \text{const} \cdot \mu_{g'}$, where const $> 0$ and $g'$ is the $G'$-orbit corresponding to $\mathcal{O}$ via the Howe-Witt theorem (Theorem 1.5).

Let $\kappa(\ ,\ )$ denote the Killing form on $g$ and define a Fourier transform by

$$\hat{\psi}(x) = \int_g \psi(y) e^{i\kappa(x, y)} \, dy \quad (\psi \in S(g), x \in g).$$

Let $\mu_{g'} \in S^*(g)$ denote the Fourier transform of $\mu_{g'}$. By Harish-Chandra, this distribution coincides with a function $\mu_{g'}(z), z \in g'$; see [W, 8.3.5]. Similarly, we have $\mu_{g'}(z'), z' \in g'$. By combining (1.6) and (1.7) with the fact that $\mu_{g'}$ is absolutely integrable against any Schwartz function, we deduce the following theorem.

THEOREM 1.8. There is a constant const $> 0$ such that for $\psi \in S(g)$ with supp $\hat{\psi} \cap \tau_{\psi}(W^{\text{max}})$ compact

$$\mu_{g'}(\psi) = \int_{g'} \psi(z') \mu_{g'}(z') \, dz' = \text{const} \int_g (\mathcal{M}(\hat{\psi}))(z) \mu_{g'}(z) \, dz,$$

where the integrals are absolutely convergent.

Now we can state our conjecture.

CONJECTURE. There is a constant const $> 0$, depending only on normalization of the Lebesgue measure on $g'$, such that

$$\tag{1.9} \mu_{\Pi'} = \mathcal{M}_0 \mu_{\Pi},$$

where $\mu_{\Pi}$ stands for the complex conjugate of the function $\mu_{\Pi}$.

Thanks to [R] and Theorems 1.5, 1.7, and 1.8, equation (1.9) would imply (1.4). Recently, we have proved that (1.9) holds in the "deep stable range" (see [DP]), where we can compute the distribution character of $\Pi'$ from that of $\Pi$. Although proving the conjecture in the general situation is at present more a matter of hard work than insight, a proof has not yet been written down.

2. The case when $(\ ,\ )$ is hermitian. In this section, $D$ is equipped with an involution $D \ni a \to \bar{a} \in D$, which is trivial only if $D = \mathbb{R}$. Let $M_{m,n}(D)$ denote the set of matrices with $m$ rows and $n$ columns and with entries from $D$. Let $M_n(D) = M_{n,n}(D)$ and let $D^n = M_{1,1}(D)$. We view $D^n$ as a left vector space over $D$ by the following formula

$$av = v \cdot \bar{a} \quad (a \in D, v \in D^n).$$

Each matrix $F \in M_n(D)$ acts on $D^n$ via left multiplication. Thus, $M_n(D)$ may be identified with $\text{End}_D(D^n)$. Since $R \subseteq D$, $D^n$ may be viewed as a real vector space,
and we have an obvious inclusion $\text{End}_R(D^n) \subseteq \text{End}_R(D^n)$. For $F \in M_n(D)$, let $\det_R(F)$ denote the determinant of $F$ viewed as an element of $\text{End}_R(D^n)$.

For two positive integers $d' \geq 2d$, set $V = D^d$ and $V' = D^{d'}$. Fix a matrix $F \in M_d(D)$ such that $F = -F^t$ and $|\det_R(F)| = 1$. Let

$$F' = \begin{pmatrix} 0 & 0 & I_d \\ 0 & F'' & 0 \\ I_d & 0 & 0 \end{pmatrix}, \quad F'' = \begin{pmatrix} I_{p'} & 0 \\ 0 & -I_{q'} \end{pmatrix},$$

$$2d + p' + q' = d', \quad p = p' + d, \quad q = q' + d.$$

Set

$$(2.1) \quad (u, v) = \overline{u}^t F v, \quad (u', v')' = \overline{u}^t F' v' \quad (u, v \in V, u', v' \in V').$$

Then $(\cdot, \cdot)$ is a nondegenerate skew-hermitian form on $V$ and $(\cdot, \cdot)'$ is a nondegenerate hermitian form on $V'$ of signature $p, q$. The corresponding isometry groups and Lie algebras can be represented in terms of matrices as follows:

$$G = \{ g \in M_d(D); \overline{g}^t F g = F \}, \quad g = \{ z \in M_d(D); \overline{z}^t F + F z = 0 \},$$

$$G' = \{ g \in M_{d'}(D); \overline{g}^t F' g = F' \}, \quad g' = \{ z \in M_{d'}(D); \overline{z}^t F' + F' z = 0 \}.$$ 

Let $W = M_{d,d'}(D)$. This is a symplectic space over $R$ with the symplectic form defined in terms of the forms (2.1) as in (1.1). Let $L' = \{ g \in G'; \overline{g}^t g = I_{d'} \}$. This is a maximal compact subgroup of $G'$. The centralizer of $L'$ in $\text{Sp}(W)$ is isomorphic to $L = G \times G$. Let $l = g \oplus g$. Then we have the moment maps $\tau_g: W \to g$, $\tau_g': W \to g'$ and $\tau_l: W \to l$ given explicitly by

$$\tau_g(w) = w F' \overline{w}^t F, \quad \tau_g'(w) = ((w \overline{w}^t + w F' \overline{w}^t) F, -(w \overline{w}^t - w F' \overline{w}^t) F),$$

$$(2.2) \quad \tau_l(w) = F' \overline{w}^t F w \quad (w \in W).$$

(These maps $\tau_g$, $\tau_g'$, $\tau_l$ are essentially determined by the fact that they are constant on the $G'$, $G$, $L'$ orbits in $W$, respectively.) We shall view $W$ as a direct sum

$$(2.3) \quad W = M_d(D) \oplus M_{d',d'-2d}(D) \oplus M_d(D),$$

where each $w \in W$ is written as $w = (a, b, c)$, $a \in M_d(D)$, $b \in M_{d',d'-2d}(D)$, $c \in M_d(D)$. In terms of (2.3), define an affine map $\sigma_g: g \to W$

$$(2.4) \quad \sigma_g(z) = \left( \frac{1}{2} z, 0, -F^{-1} \right) \quad (z \in g).$$

We shall see in (2.7) that this is a section to the map $\tau_g$. 


We shall identify the general linear group $GL(V)$ with a subgroup of $G'$ by the following injective group homomorphism

$$
(2.5) \quad \text{GL}(V) \ni g \to \begin{pmatrix} g & 0 & 0 \\ 0 & I_{d-2d} & 0 \\ 0 & 0 & (\bar{g}^{-1})^{-1} \end{pmatrix} \in G'.
$$

Then for $g \in \text{GL}(V)$ and $z \in g$

$$
(2.6) \quad g(\sigma_g(z)) = \sigma_g(z) \cdot g^{-1} = \left( \frac{1}{2} zg^{-1}, 0, -F^{-1}\bar{g} \right)
$$

and

$$
\tau_g(g(\sigma_g(z))) = z
$$

$$
(2.7) \quad \tau_g(\sigma_g(z)) = \begin{pmatrix} \frac{1}{2} zg^{-1} & 0 & -gF^{-1}\bar{g} \\ 0 & 0 & 0 \\ \frac{1}{4}(\bar{g}^{-1})^{-1}z^tFzg^{-1} & 0 & -\frac{1}{2}(\bar{g}^{-1})^{-1}z^t\bar{g} \end{pmatrix}.
$$

Before proceeding any further, we make the following observation. With the notation of (2.7), let $S = zF^{-1}$ and let $T = (\bar{F}^{-1})\bar{g}^tF^{-1}$. Then

$$
(2.8) \quad \tau_g(\sigma_g(z)) = \left( \begin{pmatrix} S & T \end{pmatrix} + S \right)F, -\left( \begin{pmatrix} S & T \end{pmatrix} - S \right)F.
$$

Let $\mathcal{H} = \{ S \in M_d(D); S = \bar{S} \}$ be the space of hermitian matrices of size $d$. Let $\mathcal{H}^+ = \{ S \in \mathcal{H}; S > 0 \}$ be the subset of positive definite matrices. For $S \in \mathcal{H}$, let $\mathcal{H}_S^+ = \{ T \in \mathcal{H}^+; T > (1/4)ST^{-1}S \}$, and let $\mathcal{H}_\pm^+_S = \{ P \in \mathcal{H}^+; P \pm S > 0 \}$.

**Lemma 2.9.** Fix $S \in \mathcal{H}$. Then the map

$$
\mathcal{H}_S^+ \ni T \to \frac{1}{4}ST^{-1}S + T \in \mathcal{H}_\pm^+_S
$$

is a bijection.
Proof. Suppose first that \( d = 1 \) and \( D = R \). Then the above statement means that for any \( s \in R \), the map

\[
\left( \frac{1}{2} |s|, +\infty \right) \ni t \mapsto \frac{1}{4} s^2 t^{-1} + t \in (|s|, +\infty)
\]

is a bijection. This is elementary.

Notice that for \( g \in GL(V) \)

\[
g \left( \frac{1}{4} ST^{-1} + T \right) \bar{g}^t = \frac{1}{4} (gS\bar{g}^t)(gT\bar{g}^t)^{-1}(gS\bar{g}^t) + gT\bar{g}^t.
\]

Hence, by the spectral theorem for hermitian matrices, we may assume that

\[
\frac{1}{2} S = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} I_r & 0 \\ 0 & -I_s \end{pmatrix}.
\]

The stabilizer of \((1/2)S\) in \(GL(V)\) (under the above action) consists of matrices of the form

\[
g = \begin{pmatrix} h & B \\ 0 & C \end{pmatrix}, \quad hE\bar{h}^t = E, \quad \det h(C) \neq 0.
\]

Suppose we know that the lemma holds if \( d = r + s \) (i.e., if \((1/2)S = E\)). Let \( d > r + s \).

We shall write \( T \in \mathcal{H}^+ \) in a block form

\[
T = \begin{pmatrix} T_1 & T_2 \\ T_2^* & T_3 \end{pmatrix}
\]

as in (2.11). Notice that \( T_3 > 0 \). Take \( B = -hT_2T_3^{-1} \) in (2.11). Then

\[
gT\bar{g}^t = \begin{pmatrix} h(T_1 - T_2 T_3^{-1} T_2^*)\bar{h}^t & 0 \\ 0 & CT_3 \bar{C}^t \end{pmatrix}.
\]

Thus, elements of \( \mathcal{H}^+ \) are diagonalizable via the action of the stabilizer of \((1/2)S\).
(This shall be verified shortly for the case \((1/2)S = E\).) Hence, by (2.10) the map
(2.9) is surjective.

Suppose \( T, T' \in \mathcal{H}^+_S \) and

\[
\frac{1}{4} ST^{-1} S + T = \frac{1}{4} ST'^{-1} S + T'.
\]
Write $T'$ in a block form as in (2.11):

$$T' = \begin{pmatrix} T'_1 & T'_2 \\ \overline{T'_2} & T'_3 \end{pmatrix}. $$

Then (2.13) implies that $T_2 = T'_2$ and $T_3 = T'_3$. Thus, the same $g$ as in (2.12) gives

$$gT'\overline{g'} = \begin{pmatrix} h(T'_1 - T'_2T'_3^{-1}\overline{T'_2})\overline{h'} & 0 \\ 0 & CT_3\overline{C}_3 \end{pmatrix}. $$

By combining (2.12–2.14), we get

$$\begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h(T'_1 - T'_2T'_3^{-1}\overline{T'_2})\overline{h'} & 0 \\ 0 & CT_3\overline{C}_3 \end{pmatrix}^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h(T'_1 - T'_2T'_3^{-1}\overline{T'_2})\overline{h'} & 0 \\ 0 & CT_3\overline{C}_3 \end{pmatrix} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} h(T'_1 - T'_2T'_3^{-1}\overline{T'_2})\overline{h'} & 0 \\ 0 & CT_3\overline{C}_3 \end{pmatrix}^{-1} \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} h(T'_1 - T'_2T'_3^{-1}\overline{T'_2})\overline{h'} & 0 \\ 0 & CT_3\overline{C}_3 \end{pmatrix}. $$

Hence, by taking the terms in the upper-left corner, we see that

$$E((T'_1 - T'_2T'_3^{-1}\overline{T'_2}))^{-1}E + (T'_1 - T'_2T'_3^{-1}\overline{T'_2})$$

$$= E((T'_1 - T'_2T'_3^{-1}\overline{T'_2}))^{-1}E + (T'_1 - T'_2T'_3^{-1}\overline{T'_2}).$$

Therefore (by our assumption that the lemma holds for $d = r + s$),

$$T_1 - T_2T_3^{-1}\overline{T_2} = T'_1 - T'_2T'_3^{-1}\overline{T'_2}. $$

Thus, $T_1 = T'_1$ and consequently $T = T'$. Hence, the map (2.9) is injective.

From now on, we assume $d = r + s$. Let $H$ denote the stabilizer of $(1/2)S = E$ in $GL(V)$. Let $\overline{B}^+ = \{b = \text{diag}(b_1, b_2, \ldots, b_d); b_1 \geq b_2 \geq \cdots \geq b_r > 0, b_{r+1} \geq b_{r+2} \geq \cdots \geq b_d > 0\}$. Let $U = \{g \in GL(V); g\overline{g'} = I_d\}$. By a well-known structure theorem for symmetric spaces [S, 7.1.3], the map

$$(2.15) \quad H \times \overline{B}^+ \times U \ni (h, b, u) \to hbu \in GL(V)$$
is surjective with the fiber $\{(hl, b, l^{-1}u); l \in \text{centralizer of } b \text{ in } H \cap U\}$. In particular, we see that the action of $H$ on $\mathcal{H}^+$ is diagonalizable. Hence, (2.10) implies that the map (2.9) is surjective.

It remains to prove the injectivity. Suppose $T = h b^2 h'$ and $T' = h'h' b'^2 h'' \in \mathcal{H}_S^+$ satisfy (2.13). Then

$$E(h b^2 h')^{-1} E + h b^2 h' = E(h'h'^2 h'')^{-1} E + h'h'^2 h'' ,$$

so

$$E(b^2)^{-1} E + b^2 = (h^{-1}h')(E(b'^2)^{-1} E + b'^2)(h^{-1}h')'.$$

But $E$ commutes with $b$ and $E^2 = I_d$. Hence,

$$(2.16) \quad b^{-2} + b^2 = (h^{-1}h')(b'^{-2} + b'^2)(h^{-1}h')'.$$

Moreover, the condition $T, T' \in \mathcal{H}_S^+$ implies that $b_1 \geq b_2 \geq \cdots \geq b_r > l$ and $b_{r+1} \geq b_{r+2} \geq \cdots \geq b_d > 1$. Notice that if $y \geq x \geq 1$, then $y + y^{-1} \geq x + x^{-1}$.

Hence, $b^{-2} + b^2 \in B^+$ and $b'^{-2} + b'^2 \in B^+$. Therefore, (2.15) and (2.16) imply that $b^{-2} + b^2 = b'^{-2} + b'^2$ and $h^{-1}h' = l$ for some $l$ in the centralizer of $b^{-2} + b^2$ in $H \cap U$. Notice that $b$ can be written in terms of $c = b^{-2} + b^2$

$$b = \sqrt{\frac{c + \sqrt{c^2 - 4}}{2}} .$$

Hence, $l$ commutes with $b$. Therefore, $T' = h'h'^2 h'' = hlb'^{-1} h' = h b^2 h' = T$. \square

**Corollary 2.17.** Let $(\mathcal{H} \times \mathcal{H}^+)^+ = \{(S, T) \in \mathcal{H} \times \mathcal{H}^+; T > (1/4)ST^{-1}S\}$. Then the map

$$(\mathcal{H} \times \mathcal{H}^+)^+ \ni (S, T) \mapsto \left(\frac{1}{4} ST^{-1}S + T + S, \frac{1}{4} ST^{-1}S + T - S\right) \in \mathcal{H}^+ \times \mathcal{H}^+$$

is a bijection.

**Proof.** Given $P, P' \in \mathcal{H}^+$, we want to show that there is a unique $(S, T) \in (\mathcal{H} \times \mathcal{H}^+)^+$ such that

$$\frac{1}{4} ST^{-1}S + T + S = P$$

$$\frac{1}{4} ST^{-1}S + T - S = P'.$$
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Clearly, $S = (1/2)(P - P')$. Notice that $P + P' \pm (P - P') > 0$. Thus, $(1/2)(P + P') \in \mathcal{H}_\pm^+$. Hence, by (2.9) there is a unique $T \in \mathcal{H}_3^+$ such that $(1/4)ST^{-1}S + T = (1/2)(P + P')$. \(\square\)

Define a representation $\rho$ of $GL(V)$ on the real vector space $\mathcal{H}$ by

\[
\rho(g)S = gSg^t \quad (g \in GL(V), S \in \mathcal{H}).
\]

**Corollary 2.19.** Let $dP$ denote a Lebesgue measure on $\mathcal{H}$. Then there is $\text{const} > 0$ such that for a test function $\psi$

\[
\int_{\mathcal{H}^+} \int_{\mathcal{H}^+} \psi(P, P') dP' dP = \text{const} \int_{(\mathcal{H} \times \mathcal{H}^+)^+} \psi \left( \frac{1}{4} ST^{-1}S + T + S, \frac{1}{4} ST^{-1}S + T - S \right) \left| \det_{\mathbb{R}} \left( 1 - \rho \left( \frac{1}{2} ST^{-1} \right) \right) \right| dT dS.
\]

**Proof.** The derivative of the map (2.17) at $(S, T)$ coincides with the following linear map:

\[
(\Delta S, \Delta T) \rightarrow \left( \frac{1}{4} \Delta S \cdot T^{-1}S + \frac{1}{4} S \cdot T^{-1} \cdot \Delta S - \frac{1}{4} ST^{-1} \cdot \Delta T \cdot T^{-1}S + \Delta T + \Delta S, \right.
\]

\[
\left. \frac{1}{4} \Delta S \cdot T^{-1}S + \frac{1}{4} S \cdot T^{-1} \cdot \Delta S - \frac{1}{4} ST^{-1} \cdot \Delta T \cdot T^{-1}S + \Delta T - \Delta S \right).
\]

After a linear transformation, the right-hand side becomes

\[
\left( \Delta S, \frac{1}{4} \Delta S \cdot T^{-1}S + \frac{1}{4} S \cdot T^{-1} \cdot \Delta S - \frac{1}{4} ST^{-1} \cdot \Delta T \cdot T^{-1}S + \Delta T \right).
\]

Hence, the determinant of the above is a constant multiple of the determinant of the following linear map:

\[
\Delta T \rightarrow \Delta T - \frac{1}{4} ST^{-1} \cdot \Delta T \cdot T^{-1}S = \left( 1 - \rho \left( \frac{1}{2} ST^{-1} \right) \right) \Delta T. \quad \square
\]

The following lemma is well known, but for completeness we include a simple proof.

**Lemma 2.20.** Let $r = 2 \dim_{\mathbb{R}} \mathcal{H} / \dim_{\mathbb{R}} V$ (notice here that $\dim_{\mathbb{R}} \mathcal{H} = \dim_{\mathbb{R}} g$). Then, for $d' \geq d$, there is $\text{const} > 0$ such that for a test function $\psi$

\[
\int_{M_{d',d}(D)} \psi(x\mathbb{X}^t) dx = \text{const} \int_{\mathcal{H}^+} \psi(P) |\det_{\mathbb{R}} P|^{(d' - r)/2} dP.
\]
Proof. Let $g \in GL(V)$. Then
\[
\int_{M_{d,d}(D)} \psi(gx\bar{x}^t\bar{g}^t) \, dx = |\det_{\mathbb{R}} g|^{-d'} \int_{M_{d,d}(D)} \psi(x\bar{x}^t) \, dx
\]

and
\[
\int_{\mathcal{N}^+} \psi(g^t\bar{g}^t)|\det_{\mathbb{R}} g|^{(d'-r)/2} \, dP
\]
\[
= \int_{\mathcal{N}^+} \psi(P)|\det_{\mathbb{R}} g^{-1}P(\bar{g}^{-1})^{t}|^{(d'-r)/2}|\det_{\mathbb{R}} g|^{-r} \, dP
\]
\[
= |\det_{\mathbb{R}} g|^{-d'} \int_{\mathcal{N}^+} \psi(P)|\det_{\mathbb{R}} P|^{(d'-r)/2} \, dP. \quad \square
\]

Lemma 2.21. Let
\[
(g \times GL(V))^+ = \{(z, g) \in g \times GL(V); 4I_d > (\bar{g}^t)^{-1}Fzg^{-1}((\bar{g}^t)^{-1}Fzg^{-1})^t\}.
\]
Set
\[
M(z, g) = \left| \det_{\mathbb{R}} \left( \frac{1}{4}(\bar{g}^t)^{-1}Fzg^{-1}((\bar{g}^t)^{-1}Fzg^{-1})^t + 1 - (\bar{g}^t)^{-1}Fzg^{-1} \right) \right|^{(q-r)/2}
\]
\[
\times \left| \det_{\mathbb{R}} \left( \frac{1}{4}(\bar{g}^t)^{-1}Fzg^{-1}((\bar{g}^t)^{-1}Fzg^{-1})^t + 1 + (\bar{g}^t)^{-1}Fzg^{-1} \right) \right|^{(q-r)/2}
\]
\[
\times \left| \det_{\mathbb{R}} \left( 1 - \rho \left( \frac{1}{2}(\bar{g}^t)^{-1}Fzg^{-1} \right) \right) \right| |\det_{\mathbb{R}}(g)|^{d'-r}.
\]
One can normalize all the measures involved so that for a test function $\phi \in C_c(W^{\max})$
\[
\int_W \phi(w) \, dw = \int_{(g \times GL(V))^+} \int_{L'} \phi(kg(\sigma_0(z))) \, dk \, M(z, g) \, dz \, dg.
\]
Proof. Define a function $\psi$ on $L$ by
\[
\psi \circ \tau_{\iota}(w) = \int_{L'} \phi(wk) \, dk.
\]
Since there is a matrix $u$ such that $u\bar{u}^t = I_d$ and $uF\bar{u}^t = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix}$, (2.2) and
(2.20) imply that

\[ \int_{\omega} \phi(w) \, dw = \int_{\omega} \psi \circ \tau_{i}(w) \, dw \]

\[ = \text{const} \int_{M_{\delta, p}(D)} \int_{M_{\delta, q}(D)} \psi(x \bar{\xi} F, -y \bar{\eta} F) \, dy \, dx \]

\[ = \text{const} \int_{\mathcal{H}^+} \int_{\mathcal{H}^+} \psi(PF, -P'F) |\text{det} \mathbb{R} P^{(p-r)/2} | \text{det} \mathbb{R} P'^{(q-r)/2} \, dP' \, dP. \]

By (2.19), the above is a constant multiple of

\[ \int_{(\mathcal{H} \times \mathcal{H})^+} \psi \left( \left( \frac{1}{4} ST^{-1} S + T + S \right) F, -\left( \frac{1}{4} ST^{-1} S + T - S \right) F \right) \]

\[ \times \left| \text{det} \mathbb{R} \left( \frac{1}{4} ST^{-1} S + T + S \right) \right|^{(p-r)/2} \]

\[ \times \left| \text{det} \mathbb{R} \left( \frac{1}{4} ST^{-1} S + T - S \right) \right|^{(q-r)/2} \]

\[ \times \left| \text{det} \mathbb{R} \left( 1 - \rho \left( \frac{1}{2} ST^{-1} \right) \right) \right| \, dT \, dS. \]

Let us write \( S = zF^{-1} \) and \( T = (F^{-1})' \bar{g}'gF^{-1} \), as in (2.8). Then again by (2.20), the above is a constant multiple of

\[ \int_{(g \times GL(V))^+} \psi(\tau_{i}(g(\sigma_{\delta}(z)))) \left| \text{det} \mathbb{R} \left( \frac{1}{4} zg^{-1}(\bar{g}')^{-1} z'^{t} + (F^{-1})^{-1} \bar{g}'gF^{-1} + zF^{-1} \right) \right|^{(p-r)/2} \]

\[ \times \left| \text{det} \mathbb{R} \left( \frac{1}{4} zg^{-1}(\bar{g}')^{-1} z'^{t} + (F^{-1})^{-1} \bar{g}'gF^{-1} - zF^{-1} \right) \right|^{(q-r)/2} \]

\[ \times \left| \text{det} \mathbb{R} \left( 1 - \rho \left( \frac{1}{2} zg^{-1}(\bar{g}')^{-1} F'^{t} \right) \right) \right| \left| \text{det} \mathbb{R} g \right|' \, dg \, dz. \]

Using the relation

\[ \text{det} \mathbb{R} (A + (F')^{-1} \bar{g}'gF^{-1}) = \text{det} \mathbb{R} (F'(\bar{g}')^{-1} AFg^{-1} + 1)|\text{det}(g)|^2, \]

one can transform (2.22) to obtain the integral formula of (2.21). \( \square \)
Finally, we make a specific choice of the matrix $F$:

$$ F = \begin{cases} 
(0 & I_{d/2} \\
-I_{d/2} & 0) & \text{if } D = R \\
(i_{d} & 0) & \text{if } D = C \\
0 & 0 - i_{d+s} \\
i_{d} & \text{if } D = H. 
\end{cases} $$

Also, let

$$ B^+ = \begin{cases} 
\{b = \text{diag}(b_1, \ldots, b_d); b_1 = b_{d/2+1} > b_2 \\
\quad \quad = b_{d/2+2} > \cdots > b_{d/2} = b_d > 0\} & \text{if } D = R \\
\{b = \text{diag}(b_1, \ldots, b_d); b_1 > b_2 > \cdots > b_s > 0, \\
\quad \quad b_{s+1} > b_{s+2} > \cdots > b_d > 0\} & \text{if } D = C \\
\{b = \text{diag}(b_1, \ldots, b_d); b_1 > b_2 > \cdots > b_d > 0\} & \text{if } D = H. 
\end{cases} $$

There is a function $\delta(b)$, $b \in B^+$, [S, 8.1.1] such that

$$ \int_{GL(V)} f(g) \, dg = \int_{U} \int_{B^+} \int_{G} f(ubh)\delta(b) \, du \, db \, dh, $$

(2.23) $\delta(b) \leq \text{const} \cdot (b_1^{d-1}b_2^{d-3} \cdots b_d^{d+1})^n, \quad n = \dim_{R}(D).$

Finally, we arrive at a precise formulation of the Theorem 1.6 (a).

**Theorem 2.24.** Let $(g \times B^+)^+ = \{(z, b) \in g \times B^+; 4I_{d} > (b^{-1}Fzb^{-1})(b^{-1}Fzb^{-1})'\}$. Let

$$ m(z, b) = \left| \det_{R} \left( \frac{1}{4} b^{-1}Fzb^{-1} \left( b^{-1}Fzb^{-1} \right)' + 1 - b^{-1}Fzb^{-1} \right) \right|^{(p-r)/2} $$

$$ \times \left| \det_{R} \left( \frac{1}{4} b^{-1}Fzb^{-1} \left( b^{-1}Fzb^{-1} \right)' + 1 + b^{-1}Fzb^{-1} \right) \right|^{(q-r)/2} $$

$$ \times \left| \det_{R} \left( 1 - b^{-1}Fzb^{-1} \right) \right| \left| \det_{R} b \right|^{d-r} \delta(b). $$

Then, with appropriate normalization of all the measures involved,

$$ \int_{W} \phi(w) \, dw = \int_{G} \int_{(0 \times B^+)^+} \int_{L'} \phi(h \cdot \sigma_{g}(z) \cdot b^{-1}k^{-1}) \, dk \, m(z, b) \, db \, dz \, dh. $$

**Proof.** We apply the equation (2.23) to (2.21) by writing $g = ubh$ and then changing $z$ to $h^{-1}zh$. \qed
Proof of Theorem 1.6. It remains to show Theorem 1.6 (b) and the last statement. Let \( \psi \) be a continuous rapidly decreasing function on \( g' \). Then

\[
\mathcal{A} \psi(z) = \int_{(g \times B^+)^+} \int_{L'} \psi \circ \tau_g (\sigma_g(z)b^{-1}k^{-1}) \, dk \, m(z, b) \, db.
\]

Since the function \( m \) is bounded on \((g \times B^+)^+\), it is clear from (2.7) that

\[
\mathcal{A} \psi(z) \leq \operatorname{const} \int_{B^+} \int_{L'} |\psi \circ \tau_g (\sigma_g(z)b^{-1}k^{-1})| |\det_{\mathbb{R}} b|^{d'-r} \delta(b) \, dk \, db
\]

\[
\times \operatorname{const}_N \int_{B^+} (1 + |zb^{-1}|)^{-N} (1 + |bF^{-1}b|)^{-N} |\det_{\mathbb{R}} b|^{d'-r} \delta(b) \, db.
\]

Notice that

\[
|zb^{-1}|^2 = |bFzb^{-1}|^2 = \sum_{i,j} b_i^2 |(Fz)_{i,j}|^2 b_j^{-2}
\]

\[
= \sum_i |(Fz)_{i,i}|^2 + \sum_{i<j} |(Fz)_{i,j}|^2 (b_i^2 b_j^{-2} + b_i^{-2} b_j^2)
\]

\[
\geq \sum_i |(Fz)_{i,i}|^2 + 2 \sum_{i<j} |(Fz)_{i,j}|^2 = |Fz|^2 = |z|^2.
\]

Further, the inequality (2.23) implies that for \( N > 0 \) large enough

\[
(2.26) \quad \int_{B^+} (1 + |bF^{-1}b|)^{-N} |\det_{\mathbb{R}} b|^{d'-r} \delta(b) \, db < \infty.
\]

Thus,

\[
|\mathcal{A} \psi(z)| \leq \operatorname{const}_N (1 + |z|)^{-N} \quad (z \in g).
\]

This verifies Theorem 1.6 (b).

If \( \psi \in C^\infty(g') \) and \( \operatorname{supp} \psi \cap \tau_g (W^{\max}) \) is compact, then we integrate over a compact subset of \((g \times B^+)^+\) in (2.25). The projection of this set on \( B^+ \) is also compact. Thus, we may take derivatives with respect to \( z \in g \) and estimate as above without appealing to the inequality (2.26). Hence, the last statement follows. \( \square \)

3. The case when \((, , ) \) is skew-symmetric and \( D = \mathbb{R} \) or \( \mathbb{C} \). In this section, \( D \) is equipped with the trivial involution. Let \( d' \geq 2d \) be positive integers with \( d' \) even. Let \( V = D^d \) and let \( V' = D^{d'} \). Fix a nonsingular matrix \( F \in M_d(\mathbb{R}) \) such that
$F = F' = F^{-1}$. Let

$$F' = \begin{bmatrix} 0 & 0 & I_d \\ 0 & F'' & 0 \\ -I_d & 0 & 0 \end{bmatrix}, \quad F'' = \begin{bmatrix} 0 & I_{d/2-d} \\ -I_{d/2-d} & 0 \end{bmatrix}.$$

Set

$$\langle u, v \rangle = u'^* F v, \quad \langle u', v' \rangle = u'^* F' v' \quad (u, v \in V; u', v' \in V').$$

Then $\langle , \rangle$ is a nondegenerate symmetric form on $V$, and $\langle , \rangle'$ is a nondegenerate skew-symmetric form on $V'$. The corresponding isometry groups and Lie algebras can be represented in terms of matrices as follows:

$$G = \{ g \in M_d(D); g'^* F g = F \}, \quad g = \{ z \in M_d(D); z'^* F + F z = 0 \},$$

$$G' = \{ g \in M_d(D); g'^* F' g = F' \}, \quad g' = \{ z \in M_d(D); z'^* F' + F' z = 0 \}.$$

Let $L' = \{ g \in G'; \overline{g} g = I_d \}$, where $g \rightarrow \overline{g}$ is the complex conjugation if $D = \mathbb{C}$, and is trivial otherwise. This is a maximal compact subgroup of $G'$. Let us view the quaternions as matrices

$$\begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, \quad a, b \in \mathbb{C}.$$

Let $j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. We identify

$$\mathbb{C} \ni a \rightarrow \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} \in \mathbb{H}.$$

Then

$$\mathbb{H} = \mathbb{C} \oplus \mathbb{C} j, \quad aj = j \overline{a}, a \in \mathbb{C}.$$

Let

$$D' = \begin{cases} \mathbb{C} & \text{if } D = \mathbb{R} \\ \mathbb{H} & \text{if } D = \mathbb{C}. \end{cases}$$

If $D' = \mathbb{H}$, let $j \in D'$ be as in (3.3). If $D' = \mathbb{C}$, let $j \in D'$ be $\sqrt{-1}$. Then

$$D' = D \oplus D j.$$
Let $D' \ni a \to \bar{a} \in D'$ denote the standard nontrivial involution over $\mathbb{R}$. Let
\[
L = \{ g \in M_d(D'); \overline{g}^t jFg = jF \}, \quad \text{and}
\]
\[
I = \{ x \in M_d(D'); \overline{x}^t jF + jFx = 0 \}.
\]
Notice that if $G \cong O_{p,q}$, then $L \cong U_{p,q}$, and if $G \cong O_p(\mathbb{C})$, then $L \cong O_{2p}^\ast$.

Let $W = M_{d,d}(D)$. The groups $G, G'$ act on $W$ as indicated in the introduction. The moment maps $\tau_g : W \to g$, $\tau_i : W \to I$ and $\tau_{g'} : W \to g'$ are given explicitly by
\[
\tau_g(w) = wF'w'F, \quad \tau_i(w) = (wF'w + w\bar{w}'j)F, \quad \tau_{g'}(w) = F'wFw \quad (w \in W).
\]
As in (2.3), we have a decomposition
\[
W = M_{d}(D) \oplus M_{d,d-2d}(D) \oplus M_{d}(D)
\]
in terms of which we define a section of the map $\tau$:
\[
\sigma(z) = \begin{pmatrix}
1/2 & z & 0 & F^{-1}
\end{pmatrix} \quad (z \in g).
\]
We identify $GL(V)$ with a subgroup of $G'$ by a formula analogous to (2.5):
\[
GL(V) \ni g \to \begin{pmatrix}
g & 0 & 0 \\
0 & I_{d-2d} & 0 \\
0 & 0 & (g')^{-1}
\end{pmatrix} \in G'.
\]
Then, for $g \in GL(V)$ and $z \in g$,
\[
g(\sigma(z)) = \sigma(z) \cdot g^{-1} = \begin{pmatrix}
1/2 & zg^{-1} & 0 & F^{-1}g'
\end{pmatrix}.
\]
With the notation of (3.8), we have
\[
\tau_g(g(\sigma(z))) = z
\]
\[
\tau_i(g(\sigma(z))) = \begin{pmatrix}
1/4 & zg^{-1}(\overline{g}')^{-1}z' + Fg'\overline{g} - zgF
\end{pmatrix} jF
\]
\[
\tau_{g'}(g(\sigma(z))) = \begin{pmatrix}
1/2 & g'g^{-1} & 0 & gFg'
0 & 0 & 0 \\
0 & 0 & 0 \\
-1/4(g')^{-1}z'Fg^{-1} & 0 & -1/2(g')^{-1}z'g'
\end{pmatrix}.
With the notation of (3.9), let \( S = zF \) and let \( T = Fg^t\overline{g}F \). Then,

\[
\tau_i(g(\sigma_0(z))) = \left( \frac{1}{4} (Sj)T^{-1}(\overline{S})^t + T - Sj \right) jF = \left( \frac{1}{4} S(\overline{T})^{-1}\overline{S_t} + T - Sj \right) jF,
\]

(3.10)

\[ S = -S', \quad T = \overline{T}', \quad T > 0. \]

Let \( \mathcal{S} = \{ S \in M_d(D) ; S = -S' \} \) be the space of skew-symmetric matrices of size \( d \). Let \( \mathcal{H}^+ = \mathcal{H}^+(D) \) be the set of positive hermitian matrices of size \( d \) as before. Similarly, we define \( \mathcal{H}^+(D') \).

**Lemma 3.11.** Let \( (\mathcal{S} \times \mathcal{H}^+)^+ = \{(S, T) \in \mathcal{S} \times \mathcal{H}^+ ; T > (1/4)\overline{ST^{-1}}\overline{S_t} \} \). Then the map

\[
(\mathcal{S} \times \mathcal{H}^+)^+ \ni (S, T) \mapsto \frac{1}{4} \overline{ST^{-1}}\overline{S_t} + T - Sj \in \mathcal{H}(D')^+
\]

is a bijection.

**Proof.** The group \( GL(V) = GL(D^d) \subseteq GL(D^d) \) acts on \( \mathcal{S}, \mathcal{H}(D), \mathcal{H}(D') \) by

\[
g(S) = gSg^t, \quad g(T) = gTg^t, \quad g(P) = gPg^t \quad (g \in GL(V), S \in \mathcal{S}, T \in \mathcal{H}, P \in \mathcal{H}(D')).
\]

Moreover, we have the following formula:

\[
g \left( \frac{1}{4} \overline{ST^{-1}}\overline{S_t} + T - Sj \right) g^t = \frac{1}{4} (gSg')(\overline{gTg'})^{-1}(gSg')^t + (gTg') - (gSg')j.
\]

Clearly, the action of \( GL(V) \) on \( \mathcal{S} \times \mathcal{H}^+ \) preserves \( (\mathcal{S} \times \mathcal{H}^+)^+ \). Fix \( S \in \mathcal{S} \). Given \( P \in \mathcal{H}^+(D') \) such that \( P - Sj \in \mathcal{H}^+(D') \), we will show that there is a unique \( T \in \mathcal{H}^+(D') \) such that

\[
(3.12) \quad \frac{1}{4} \overline{ST^{-1}}\overline{S_t} + T - Sj = P.
\]

Using the action of \( GL(V) \), we may assume that

\[
\frac{1}{2} S = \begin{pmatrix} E & 0 \\ 0 & 0 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.
\]

The stabilizer of \( (1/2)S \) in \( GL(V) \) consists of matrices of the form

\[
g = \begin{pmatrix} h & B \\ 0 & C \end{pmatrix}, \quad hEh^t = E, \ det(C) \neq 0.
\]
As in the proof of (2.9), we may reduce to the case \( (1/2)S = E \) and complete the proof.

For \( g \in GL(V) \) and \( T \in \mathcal{H} \), set

\[
\rho(g)T = g\overline{T}g'.
\]

Then \( \rho(g) \in \text{End}_R(\mathcal{H}) \). As in (2.19), one can verify the following lemma.

**Lemma 3.13.** One can normalize the Lebesgue measures \( dQ, dT, dS \) on \( \mathcal{H}(D') \), \( \mathcal{L}, \mathcal{H}(D) \), respectively, so that for a test function \( \psi \)

\[
\int_{\mathcal{H}+}(D') \psi(Q) dQ = \int_{(\mathcal{L} \times \mathcal{H}^+)^+} \psi \left( \frac{1}{4} S(\overline{T})^{-1} (\overline{S})' + T - Sj \right)
\times \left| \text{det}_R \left( 1 - \rho \left( \frac{1}{2} S(\overline{T})^{-1} \right) \right) \right| dT dS.
\]

**Lemma 3.14.** Let

\[
(g \times GL(V))^+ = \{(z, g) \in g \times GL(V); 4I_d > ((g')^{-1}Fzg^{-1})((g')^{-1}Fzg^{-1})' \}.
\]

Let \( r' = 2 \dim_R \mathcal{H}(D')/\dim_R(D^d) \) and let \( r = 2 \dim_R \mathcal{H}(D)/\dim_R(D^d) \). Set

\[
M(z, g) = \left| \text{det}_R \left( \frac{1}{4} (g')^{-1}Fzg^{-1}((g')^{-1}Fzg^{-1})' + 1 - (g')^{-1}Fzg^{-1}Fj \right) \right|^{d'/2-r'}
\times \left| \text{det}_R \left( 1 - \rho \left( \frac{1}{2} (g')^{-1}Fzg^{-1} \right) \right) \right| \left| \text{det}_R(g) \right|^{d' - 2r' + r'}.
\]

One can normalize all the measures involved so that for a test function \( \phi \in C_c(W'^{\text{max}}) \),

\[
\int W \phi(w) dw = \int_{(g \times GL(V))^+} \int_{L'} \phi(kg(\sigma_{\phi}(z))) dk M(z, g) dz dg.
\]

**Proof.** Define a function \( \psi \) on \( L' \) by

\[
\psi(w) = \int_{L'} \phi(wk) dk.
\]

Then by (3.5) and (3.13),

\[
\int W \phi(w) dw = \int W \psi \circ \tau_1(w) dw
\]

\[
\int W \psi((wF'w'j + w\overline{w}'jF) dw = \int W \psi(w(1 - F'j)\overline{w}'jF) dw.
\]
Set \( D = \begin{pmatrix} 0 & I_{d'/2-d} \\ I_d & 0 \end{pmatrix} \) so that \( F' = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix} \); see (3.1). There is an isomorphism of vector spaces

\[
M_{d,d'}(D) = M_{d,d'/2}(D) \oplus M_{d',d/2}(D) \ni w = (A, B) \rightarrow A + BDj \in M_{d,d'/2}(D'),
\]

and the formula

\[
w(1 - F'j)\bar{w}' = (A + BDj)(\bar{A} + BD\bar{j})'.
\]

Thus, by Lemmas 2.20 and 3.13:

\[
\int \phi(w) \, dw = \int_{M_{d,d'/2}(D')} \psi(x\bar{x}'jF) \, dx
\]

\[
= \int_{\mathfrak{g}^+(D')} \psi(PjF) |\det_R P|^{(d'/2-r)/2} \, dP
\]

\[
= \int_{(g \times \mathfrak{g}^+)^+} \psi \left( \left( \frac{1}{4} S(\bar{T})^{-1} \bar{S} + T - Sj \right) jF \right)
\]

\[
\times \left| \det_R \left( \frac{1}{4} S(\bar{T})^{-1} \bar{S} + T - Sj \right) \right|^{(d'/2-r)/2}
\]

\[
\times \left| \det_R \left( 1 - \rho \left( \frac{1}{2} S(\bar{T})^{-1} \right) \right) \right| \, dT \, dS.
\]

If we write \( S = zF \) and \( T = Fg'\bar{g}F \) as in (3.10), then, again by Lemma 2.20,

\[
\int \phi(w) \, dw = \int_{(g \times GL(V))} \psi \left( \left( \frac{1}{4} zg^{-1}(\bar{g}')^{-1} z' + Fg'\bar{g}F - zFj \right) jF \right)
\]

\[
\times \left| \det_R \left( \frac{1}{4} zg^{-1}(\bar{g}')^{-1} z' + Fg'\bar{g}F - zFj \right) \right|^{(d'/2-r)/2}
\]

\[
\times \left| \det_R \left( 1 - \rho \left( \frac{1}{2} zg^{-1}(\bar{g}')^{-1} F \right) \right) \right| |\det_R g'|^r \, dg \, dz,
\]

and the lemma follows. □

Let us make a specific choice

\[
F = \begin{cases} 
I_p \quad & \text{if } D = R \\
0 \quad & \\
0 \quad & \\
I_q \quad & \\
I_d \quad & \text{if } D = C.
\end{cases}
\]
Let

\[ B^+ = \begin{cases} \{ b = \text{diag}(b_1, \ldots, b_d); b_1 > \cdots > b_p > 0, b_{p+1} > \cdots > b_d > 0 \} & \text{if } D = \mathbb{R} \\ \{ b = \text{diag}(b_1, \ldots, b_d); b_1 > \cdots > b_d > 0 \} & \text{if } D = \mathbb{C} \end{cases} \]

There is a function \( \delta(b), b \in B^+, [S, 8.1.1] \) such that

\[
\int_{GL(V)} f(g) \, dg = \int_{B^+} \int_{G} \int_{B^+} f(ubh) \delta(b) \, du \, db \, dh, \quad \text{and} \quad \delta(b) \leq \text{const} \cdot (b_1^{d-1}b_2^{d-3} \cdots b_d^{-d+1})^n, \quad n = \text{dim}_R(D).
\]

Finally, we arrive at a precise formulation of Theorem 1.6 (a), which can be verified the same way as Theorem 2.24.

**Theorem 3.15.** Let \( (g \times B^+)^+ = \{(z, b) \in g \times B^+; 4I_d > (b^{-1}Fzb^{-1})(\overline{b^{-1}Fzb^{-1}})^t\}. \) Let

\[
m(z, b) = \left| \det_{\mathbb{R}} \left( \frac{1}{4} b^{-1}Fzb^{-1}(\overline{b^{-1}Fzb^{-1}})^t + 1 - b^{-1}Fzb^{-1}Fj \right) \right|^{(d'/2-r')/2} \times \left| \det_{\mathbb{R}} \left( 1 - \rho \left( \frac{1}{2} b^{-1}Fzb^{-1} \right) \right) \right| \left| \det_{\mathbb{R}} b \right|^{d'-2r'} \delta(b).
\]

Then,

\[
\int_{W} \phi(w) \, dw = \int_{G} \int_{(g \times B^+)^+} \int_{L'} \phi(h \cdot \sigma_g(z) \cdot b^{-1}k^{-1}) \, dk \, m(z, b) \, db \, dz \, dh.
\]

As before,

\[
(3.16) \quad \mathcal{A}\psi(z) = \int_{(g \times B^+)^+} \int_{L'} \psi \circ \tau_{g}(\sigma_g(z)b^{-1}k^{-1}) \, dk \, m(z, b) \, db.
\]

Hence, the proof of the remaining statements of Theorem 1.6 is the same as in the previous case.

**References**


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