THE WAVE FRONT SET AND
THE ASYMMPTOTIC SUPPORT FOR p-ADIC GROUPS

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We prove that for p-adic groups the notion of the wave front set of
a representation coincides with the notion of the asymptotic support.

1. The wave front sets of finite sums of homogeneous distributions.

Let \( \Omega \) be a \( p \)-adic field of characteristic zero, with valuation \(| \cdot |\). Let \( g \)
be a finite dimensional vector space over \( \Omega \). Fix a non-trivial character
\( \chi \) of the additive group \( \Omega \), and a non-degenerate symmetric bilinear
form \( \beta \) on \( g \) with values in \( \Omega \).

For \( f \in C_c^\infty(g) \) (compactly supported, locally constant functions on
\( g \)) define a Fourier Transform by

\[
\hat{f}(Y) = \int_g \chi(\beta(Y,X))f(X)\,dX \quad (Y \in g).
\]

Here \( dX \) is a Haar measure on the additive group of \( g \) (normalized so
that the formula \( (\hat{f})^{-1}(x) = f(-x) \) holds). Then \( f \rightarrow \hat{f} \) is a bijective
mapping of \( C_c^\infty(g) \) onto itself (see [Hal] or [W, p. 107]). If \( T \) is a
distribution \( g \) then its Fourier transform \( \hat{T} \) is given by

\[
\hat{T}(f) = T(\hat{f}) \quad (f \in C_c^\infty(g)).
\]

Let \( n = \dim_\Omega(g) \). For \( f \in C_c^\infty(g) \) define

\[
f_\lambda(X) = |\lambda|^{-n}f(\lambda^{-1}X) \quad (X \in g, \ \lambda \in \Omega^\times).
\]

Fix an open subgroup \( \Lambda \) of \( \Omega^\times \) with \([\Omega^\times : \Lambda] < \infty\).

**DEFINITION 1.4.** A distribution \( T \) on \( g \) is \( \Lambda \)-homogeneous of degree
\( d \in \mathbb{C} \) if

\[
T(f_\lambda) = |\lambda|^dT(f) \quad (f \in C_c^\infty(g), \ \lambda \in \Lambda).
\]

Notice that

\[
(f_\lambda)^{-1} = |\lambda|^{-n}(\hat{f})_\lambda^{-1} \quad (f \in C_c^\infty(g), \ \lambda \in \Omega^\times),
\]

so that if \( T \) is \( \Lambda \)-homogeneous of degree \( d \) then \( \hat{T} \) is a \( \Lambda \)-homogeneous
of degree \( -n - d \). Clearly if \( T \) is a function:

\[
T(f) = \int_g T(X)f(X)\,dX,
\]
then \( T \) is \( \Lambda \)-homogeneous of degree \( d \) iff for any \( \lambda \in \Lambda \),

\[
T(\lambda X) \, dX = |\lambda|^d T(X) \, dX.
\]

The reader may safely focus on the case \( \Lambda = \Omega^\times \). In order to justify the generality of Definition 1.4 we mention that a distribution homogeneous with respect to a quasicharacter of \( \Omega^\times \) is \( \Lambda \)-homogeneous for a suitable \( \Lambda \) (see for example [G-G-PS, Ch. II]).

By fixing a base of \( \mathfrak{g} \) we can identify it with \( \Omega^n \) and use the norm

\[
(\lambda, \lambda_2, \cdots, \lambda_n) = \max\{|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|\}.
\]

The following simple fact will be used later.

**Lemma 1.7.** Let \( F \) and \( V \) be open-compact subsets of \( \mathfrak{g} \). Then there is \( \delta > 0 \) such that for any \( \lambda \in \Omega \) with \( |\lambda| < \delta \) the following inclusion holds:

\[
\lambda F + V \subseteq V.
\]

It is known that any compactly supported distribution on \( \mathfrak{g} \) has a locally constant function as a Fourier Transform.

We are going to use (1.2) to analyze the singularities of \( T \) near zero.

**Definition 1.8 ([He] §2).** A distribution \( T \) on \( \mathfrak{g} \) is \( \Lambda \)-smooth at \( Y_0 \in \mathfrak{g}\setminus\{0\} \) if there is an open neighborhood \( W \) of 0 and an open neighborhood \( V \) of \( Y_0 \) such that for any \( f \in C_\infty^c(W) \) there is \( N > 0 \) for which \( \lambda \in \Lambda \) and \( |\lambda| > N \) imply

\[
(fT) \, ^\sim (\lambda Y) = 0 \quad \text{for any } Y \in V.
\]

The complement of the set of \( \Lambda \)-smooth points of \( T \) in \( \mathfrak{g}\setminus\{0\} \) is called the \( \Lambda \)-wave front set of \( T \) at zero and is denoted \( \mathrm{WF}_{\lambda}^0(T) \).

The function \((fT) \, ^\sim\), (1.9), is sometimes called a localized Fourier Transform of \( T \) (because \( \operatorname{supp}(fT) \subseteq \operatorname{supp}(f) \)). Of course this function can be expressed in terms of the convolution

\[
(fT) \, ^\sim = \hat{f} * \hat{T}, \quad \text{where for } X, Y \in \mathfrak{g},
\]

\[
\hat{f} * \hat{T}(X) = \hat{T}(L_X \hat{f}), \quad L_X \hat{f}(Y) = \hat{f}(X - Y).
\]

Using (1.10) and the notion of a lattice in \( \mathfrak{g} \) [W, p. 28] we rephrase the Definition 1.8. For a subset \( U \subseteq \mathfrak{g} \), let \( f_U \) denote the characteristic function of \( U \).
Lemma 1.11. Let $T$ be a distribution on $\mathfrak{g}$ and let $V$ be an open-compact subset of $\mathfrak{g}\{0\}$. Then the following conditions on $V$ are equivalent:

(a) $V \cap WF^0_\Lambda(T)$ is empty.
(b) There is a lattice $U$ in $\mathfrak{g}$ and a constant $c > 0$, such that
$$f_U \ast \hat{T}(\lambda Y) = 0 \quad \text{for} \quad \lambda \in \Lambda, \ |\lambda| > c, \ Y \in V.$$
(c) There is a lattice $W$ in $\mathfrak{g}$ and for any constant $1 > \epsilon > 0$ a constant $c_\epsilon > 0$ such that for any $f \in C^\infty_c(W)$,
$$(f_\gamma T)^\wedge(\lambda Y) = 0 \quad \text{for} \quad \lambda, \gamma \in \Lambda, \ |\lambda| > c_\epsilon, \ \epsilon < |\gamma| < 1, \ Y \in V.$$

Proof. Clearly $(*)$ implies (a). The equivalence of (a) and (b) was shown by Heifetz [He, Lemma 2.2]. We shall recall his proof to see that (b) implies $(*)$. Let $W$ be the lattice dual to $U$, $f \in C^\infty_c(W)$, and let $F = - \text{supp } \hat{f}$. Lemma 1.7 applied to the sets $F$ and $V$ provides a constant $\delta > 0$. Put $c_\epsilon = \max\{\delta^{-1}c^{-1}, c\}$. Since by (1.5) $\text{supp}(f_\gamma) = \gamma^{-1} \text{supp } \hat{f}$ we see that (under the assumptions of $(*)$)

$$(f_\gamma T)^\wedge(\lambda Y) = (f_\gamma f_W T)^\wedge(\lambda Y)$$
$$= \int_\mathfrak{g} (f_\gamma)^\wedge(Z)(f_W T)^\wedge(\lambda(-\lambda^{-1}Z + Y)) \, dZ = 0. \quad \Box$$

The reader may compare this proof with [H6, 8.1.1] to see that the analogous argument in the classical situation is more complex.

Lemma 1.11 has the following immediate

Corollary 1.12. The wave front set $WF^0_\Lambda(T)$ contains the set $A$ of those $Y \in \mathfrak{g}\{0\}$ satisfying the condition that for any lattice $U \subseteq \mathfrak{g}$ and any constant $c > 0$ there is $\lambda \in \Lambda$ with $|\lambda| > c$ such that $f_\lambda \ast \hat{T}(\lambda Y) \neq 0$.

Clearly Lemma 1.11 implies that

$$WF^0_\Lambda(T) \subseteq \Lambda \cdot \text{supp } \hat{T}.$$

Also, since for any lattice $U \subseteq \mathfrak{g}$ the support of $f_U \hat{T}$ is compact, the wave front set of $T$ is the same as that associated to the truncation $T_U$ of $T$ at infinity, defined by $\hat{T}_U = \hat{T} - f_U \hat{T}$. Therefore we have another

Corollary 1.14. The wave front set $WF^0_\Lambda(T)$ is contained in the set $B$, the intersection of all $\Lambda \cdot \text{supp } \hat{T}_U$, where $U$ varies over all lattices in $\mathfrak{g}$. 

Next we define a $p$-adic analog of the classical notion of an asymptotic cone (see [Ho, 8.1.7]). For any subset $E$ of $g\setminus\{0\}$ define its $\Lambda$-asymptotic cone to be the set

$$AC_\Lambda(E) = \left\{ \lim_{j \to \infty} \lambda_j Z_j \mid \lambda_j \in \Lambda, \lim_{j \to \infty} \lambda_j = 0, Z_j \in E \right\}.$$ 

By a $\Lambda$-conical subset of $g$ we will mean a subset closed under multiplication by elements of $\Lambda$. Then $AC_\Lambda(E)$ is a closed $\Lambda$-conical subset of $g$.

**Theorem 1.16.** For any distribution $T$ on $g$ define the sets $A$ and $B$ as in Corollaries 1.12 and 1.14 respectively. Then

$$A \subseteq WF_\Lambda^0(T) \subseteq B \subseteq AC_\Lambda(\text{supp} \hat{T}).$$

Moreover all these sets coincide if $T$ is $\Lambda$-homogeneous.

**Proof.** Only the last inclusion in (1.17) remains to be verified. It is obvious, however, if we realize that for any lattice $U$ in $g$ the support of $\hat{T}_U$ is contained in the intersection of the support of $\hat{T}$ with the complement of $U$ in $g$.

**Lemma 1.18.** For any finite sequence of real numbers $d_1 < d_2 < \cdots < d_r$ and a sequence $a_1, a_2, \ldots, a_r$ of complex numbers define the function

$$F(x) = a_1 x^{d_1} + a_2 x^{d_2} + \cdots + a_r x^{d_r}, \quad (x > 0).$$

Then either $F$ is identically equal to zero or $F$ has at most $r - 1$ zeros.

We omit the elementary proof.

**Theorem 1.19.** Let $T_1, T_2, \ldots, T_r$ be $\Lambda$-homogeneous distributions on $g$ of degrees $d_1 < d_2 < \cdots < d_r$ respectively. Put $T = T_1 + T_2 + \cdots + T_r$. then

$$WF_\Lambda^0(T) = \bigcup_{j=1}^{r} WF_\Lambda^0(T_j).$$

**Proof.** Since the wave front set of a finite sum of distributions is clearly contained in the union of the wave front sets of the summands, it will suffice to verify the inclusion

$$WF_\Lambda^0(T) \supseteq \bigcup_{j=1}^{r} WF_\Lambda^0(T_j).$$
Take $V$ disjoint with $WF^0_\Lambda(T)$ as in Lemma 1.11 (a). Then by (c)

\begin{equation}
0 = (f_\gamma T)^\sim \gamma^{-1} \lambda Y = \sum_{j=1}^r |\gamma|^d (fT_j)^\sim (\lambda Y)
\end{equation}

for $f \in C^\infty_c(W), \lambda \in \Lambda, \gamma \in \Lambda, |\gamma| > c_\varepsilon, \varepsilon < |\gamma| < 1, Y \in V$.

Choose $\varepsilon > 0$ so that there are at least $r$ elements in the set $(\varepsilon, 1] \cap \{|\gamma| \in \Lambda\}$. Then Lemma 1.18 implies that each summand in (1.21) is zero.

2. $P$-adic wave front sets of group representation. Let $G$ be a connected, reductive $\Omega$-group and $G$ the subgroup of all $\Omega$-rational points in $G$. Then $G$ with its usual topology is a locally compact, totally disconnected, unimodular group. Let $g$ be the Lie algebra of $G$. Then $g$ is a vector space over $\Omega$ of finite dimension and $G$ operates on $g$ by means of the adjoint representation. Assume that the form $\beta$ in (1.1) is $G$-invariant.

Let $\pi$ be an irreducible admissible representation of $G$ and

$$\Theta_\pi(f) = \text{tr} \pi(f) \quad (f \in C^\infty_c(G))$$

be its character.

Let $N$ be the set of all elements of $g$ which are nilpotent. Then $N$ is the union of a finite number of $G$-orbits which are called the nilpotent orbits. For all this see [Ha1], [Ha2]. Harish-Chandra [He 1, p. 180] has shown that one can choose an open neighborhood $U$ of zero in $g$ and, for each nilpotent orbit $O$, a complex constant $c_O$ such that

\begin{equation}
\Theta_\pi(\exp(X)) = \sum_0 c_O \hat{\mu}_O(X) \quad (X \in U).
\end{equation}

Here $\mu_O$ is a Radon measure on $g$ given by

$$\mu_O(f) = \int_{G/G_0} f(\text{Ad} g \cdot X_0) d g^* \quad (f \in C^\infty_c(g))$$

where $X_0 \in O$ and $G_0$ is the stabilizer of $X_0$ in $G$ (see [R]).

It follows from Theorem 1 in [R], that $\mu_O$ is a $\Omega^\times$-homogeneous distribution on $g$ of degree $d = -n + \text{dim}_\Omega(O)/2$. Therefore, via statement (1.5), $\hat{\mu}_O$ is a homogeneous distribution of degree $-\text{dim}_\Omega(O)/2$.

Let $\pi$ be an admissible representation of $G$ of finite length. Put

$$T = \Theta_\pi \cdot \exp.$$

Then (2.1) implies that

$$T = \sum_{j=1}^r T_j$$
where the \( T_j \)'s are homogeneous distributions on \( g \) of degrees \( d_j \) \( (j = 1, 2, \ldots, r) \). Explicitly

\[
T_j = \sum_{\dim \mathcal{O}/2 = -d_j} c_0 \hat{\mu}_0.
\]

Retain the above notation. Then Theorem 1.19 implies the following

**Theorem 2.2.** Let \( \pi \) be an admissible representation of \( G \) of finite length. Then

\[
WF_\Lambda^0(T) = \bigcup_{j=1}^r \text{supp} \, \hat{T}_j.
\]

The left hand side of the first equation may be thought of as the wave front set of the representation \( \pi \) (see [H], [He]) and the right hand side as the asymptotic support (see [B-V]) of \( \pi \). Recall also [He, Theorem 3.4] that for \( \pi \) unitary \( WF_\Lambda^0(T) \) coincides with the wave front set of \( \pi \) defined by the trace class operators. A statement analogous to Theorem 2.2 for the real reductive Lie groups was conjectured in [B-V] (and should hold via the inverse of the Lefschetz principle). Theorem 1.19 is true in the real case and its proof is equally easy.

**Acknowledgment.** I would like to thank Roberto Scaramuzzi for informing me about the paper [HE] of D. B. Heifetz. I am indebted to Roger Howe for the helpful revision of the original text. This text has been rewritten later by the author along the lines sketched by the referee. I would like to thank him for his suggestions.

**References**


Received February 18, 1987. Supported by NSF Grant DMS-8503781.

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